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Department of Mechanics and Mathematics

V. A. Ogorodnikov

NUMERICAL MODELLING OF DISCRETE RANDOM PROCESSES AND FIELDS

Teaching aid

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Monte Carlo Methods'

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This Teaching aid is published in the framework of master educational programme NRU NSU "Numerical Statistical Modelling and Simulation. Monte Carlo Methods" for teaching foreign students in English. It presents methods for the numerical simulation of Gaussian and non-Gaussian processes and fields of discrete argument, based on the method of conditional distributions.

1. INTRODUCTION

Methods for the numerical simulation of random processes and fields are widely used in solving theoretical and applied problems in various fields of science and technology, such as statistical meteorology, oceanography, radio physics, economics, financial mathematics, etc., and area of application of these methods and the complexity of the problems, are solved on the basis of these methods is increasing.

In statistical meteorology, climatology, oceanography and hydrology applications of these methods have long become a tradition. With these methods, a wide range of problems, including the problem of assimilation of hydrometeorological information, study the probabilistic properties of actually observed processes, probabilistic forecasting, study the properties of statistical estimates, the synthesis of dynamic and probabilistic methods for the description of real processes etc. are efficiently solved.

The starting point for the solution of these problems is to build probabilistic models of real processes and fields. Under the numerical probabilistic model of the real time series usually understand an artificial random sequence that has a certain set of pre-selected probability characteristics which coincides with the observed one. Similarly we define a probabilistic model of the real field on a regular or irregular grid or at any point of a given area.

Typical problems of this class are the problem of the emissions of the process or of the field for a given level. These problems are are directly linked with the problem the study and prediction of extreme weather conditions, and the accuracy of the results depends on the quality of the selected probability model. Investigation of these and related issues of interpretation of meteorological data based on the use of statistical modeling methods, dedicated, eg, [3,8,9,11,13,16,17,19,21,26,28,29,37].

Statistical modeling techniques can be effectively used in the tasks of calculation of dynamic impact of meteorological processes on dynamical systems, sites, structures, biological processes, etc. Here, the main interest is the study of the system response to the impact of a complex meteorological factors. As examples of the problem related to the study of deformations of high-rise buildings under wind load, cooling of the heated premises under the combined influence of temperature and wind, the study of the productivity of agricultural crops depending on the climatic variability of the complex meteorological parameters affecting the productivity of. In oceanography examples of this class of problems are the problems associated with the

study of probability laws of reservoirs level fluctuations as a function of the variability of the river flow or wind [3,37], etc. In cases where the reaction can not be expressed by input action system in a simple analytic expression, although equations describing the system are known (these equations may be non-linear), the results can be obtain with the help of repeated numerical solution of the equations for independent realizations of influence of meteorological processes constructed in accordance with a suitable probabilistic model.

Statistical modeling techniques have been widely used in solving theoretical and applied problems in statistical oceanography and are reflected in a series of monographs and articles [3,13,33]. These papers deal with a wide range of issues related to the study of the probabilistic structure of scalar and vector oceanographic processes and fields.

The concept of the description of real processes using probabilistic models developed, the issues of verification models are investigated. A wide class of methods for estimation of various characteristics of meteorological and oceanographic processes on real data is developed and investigated. The methods for modeling of vector and periodically correlated processes are also developed. On the basis of these methods and algorithms are built and verified complex probabilistic models of stationary and non-stationary oceanographic processes for different ranges of variability of oceanographic parameters.

The use of multi-dimensional stationary and non-stationary stochastic time series models of meteorological parameters demonstrated high potential of the method of statistical modeling in solving the problems of climate modeling and prediction, in the simplest case – the problems of simulation of climate scenarios and in more general cases - the problems of climate predictability and forecasting of extreme climatic situations. Statistical modeling techniques can also be effectively used in the problems of probabilistic forecasting of meteorological processes [16,19] and the related problems of optimal use of prognostic information for economic decision-making.

The accumulated experience of the probabilistic modeling of real processes and fields, as well as current trends in the development of statistical meteorology and oceanography pose new urgent problems associated with the use of statistical modeling. This is primarily environmental problems, for solving of which there is need to develop methods for complex probabilistic modeling of atmospheric and oceanographic processes engaging a large amount of real hydro-meteorological and oceanographic information. To do this, it is necessary also develop methods for combining

hydro-thermodynamic and probabilistic approaches to the description of real processes [16,26,27,28,42]. All this defines new requirements for the numerical probabilistic models - increasing the dimension of the task, involving a large amount of factual information, the account in the model of a large number of statistical parameters. All the increasing power of modern computers is helping to solve these problems.

Mathematical apparatus to solve the above problems is the method of statistical modeling and, in particular, the numerical simulation methods of random processes and fields with given probabilistic properties. At the present stage, the main characteristics used in the construction of numerical algorithms are one-dimensional distributions, correlation functions or spectral densities of of relevant processes and fields.

Modeling methods of stationary Gaussian processes and homogeneous scalar and vector fields on the basis of Gaussian spectral representation [15,25,27,31] have developed the most complete. Known methods of modeling of stationary random processes, as well as non-homogeneous random fields (Gaussian and non-Gaussian) are reduced to the use of the spectral parametric models [29,31]), in which the parameters are functions of time or space coordinates. Another important class of non-stationary processes, widely used in solving applied problems in statistical meteorology and oceanography include periodically nonstationary processes [3,11,13]. One of the main approaches to modeling processes of this class is also a spectral representation of the process, according to which the amplitude is a stationary random discrete vector process with specific correlation properties [3].

Many researchers [15,23-25,27,30,31] over the years have dealt with modeling of non-Gaussian processes and fields. One of the most common methods of construction of non-Gaussian processes and fields is reduced to a functional transformation of a Gaussian process (or field) such that the one-dimensional distribution of the process obtained as a result this transformation coincided with the given. This method is known as "Method of inverse distribution functions" for the first time proposed and investigated Z.A. Piranashvilli in [30]. When implementing this method is not always possible to find such a Gaussian process such that the one-dimensional distribution and correlation function of the simulated process would coincide with the set. The questions of compatibility of one-dimensional distributions and correlations in the framework of the "method of inverse distribution functions" have been investigated in a lot of work, for example [27,31].

Modification of the method of functional transformation of a Gaussian

process [29], taking into account the specifics of spectral models, also used in the development of non-Gaussian spectral models. Approximate modification "of the method of inverse distribution functions", designed for modeling of time series data on actual observations based on the normalization of the real number was considered in [23,37].

Well known as methods based on the use of point streams Palma [24] to simulate the non-Gaussian processes and fields with an arbitrary one-dimensional distribution and arbitrary convex correlation function. Important place among these methods take th methods of numerical simulations of Gaussian and non-Gaussian processes and fields with a discrete argument [18, 20,22,27,32,42]. These techniques and the corresponding algorithms are best suited to the construction of probabilistic models of hydrometeorological processes according to real observations.

For modeling non-stationary processes with discrete time, as well as non-homogeneous spatio-temporal fields, for example, the simulation of sea waves, are often used autoregression-moving average model with coefficients depending on the time and space coordinates.

Simulation of Gaussian processes and fields plays a fundamental role in the construction of non-Gaussian models for the reason that the class of possible correlations for the Gaussian process is the widest (it is determined by the condition of non-negative definiteness of the corresponding correlation function, or matrix), so the functional transformation of a Gaussian process allow us to describe a wide enough class of correlation functions non-Gaussian process. In the basis of simulation of Gaussian vectors and processes with discrete argument are different types of linear transformations. For example on these linear transformations are based the such model as autoregressive model, moving average model, mixed-autoregressive- moving average model. The connecting link of most methods of modeling of Gaussian processes with discrete argument, based on the linear transformations, is the method of conditional distributions [15,27,39]. Specificity and main advantage of this method is that class and block Toeplitz Toeplitz-covariance matrices, in contrast to other matrices allows one to construct a recursive algorithm simulation and thereby fundamentally increasing the number of used parameters and the dimension of the task. The class of Gaussian processes defined by Toeplitz and block-Toeplitz covariance matrices is quite wide and has a large area of applications [6,18,27]. On construction and analysis of algorithms for constructing probabilistic models of hydrometeorological processes and fields on the basis of the method of conditional expectations are devoted to the first two chapters of this manual.

In the first chapter the methods of modeling of scalar Gaussian processes with discrete argument are considered. The focus is on methods and algorithms for simulation of Gaussian random processes of discrete argument on the basis of "conditional distributions." Class of processes of this type is quite wide. It includes: a stationary random sequence scalar correlation function in any number of fixed points; stationary time series of Euclidean vectors in a given point of any region; complexes of Euclidean vectors on a given grid, which can be interpreted in applications, for example, as wind velocity vector time series , ocean currents, or the corresponding vector fields on a regular or irregular grid, stationary time series of vectors, which can be interpreted as a set of meteorological and oceanographic processes (time series of temperature, solar radiation, precipitation, etc.).

Based on the method of "conditional distributions" modeling algorithms stationary Gaussian series with the correlation matrix of Toeplitz type, and methods of regularization and control the accuracy of the algorithms are described. Conditions for the existence of stationary solutions autoregressive process with a predetermined correlation function in an arbitrary fixed number of points are given.

In the second chapter questions of numerical simulation of multidimensional Gaussian processes discrete argument with a given block-Toeplitz covariance matrix are considered. The internal structure of blocks of block-Toeplitz correlation matrix may be different. In general, they are not symmetrical and their structure depends only on the requirement that the matrix was estimated nonnegative definite. For a homogeneous field blocks are Toeplitz matrix form, for a homogeneous isotropic field, these blocks are also symmetrical.

Algorithms for simulation of multi-dimensional Gaussian processes in the vector version of the method of "conditional distributions" are described. The structure of the matrix and the specificity of this method allows to implement this method recursively. It allows to solve effectively the problem of sufficiently high dimension. Properties and range of applicability of these algorithms for constructing of probabilistic models of spatial, spatio-temporal, as well as vector fields of meteorological and oceanographic fields on regular and irregular grid investigated. Simulation algorithm periodically correlated Gaussian process with arbitrary covariance matrix of the block-Toeplitz form is given.

Using the built processes and fields as the initial data for the corresponding multivariate autoregressive model allows significantly increase the dimension of the problem to be solved.

In the third chapter, some aspects of modeling the non-Gaussian random processes and fields are considered. Examples of statistical modeling of hydrometeorological processes taking into account the specifics of one-dimensional distributions are given. For stationary Gaussian sequences based on the method of "conditional distributions" the recursive algorithms modeling "conditional realizations" of the process at specified values at equally spaced points in time are built. For the case of a spatially homogeneous Gaussian fields approximation algorithm of modeling of conditional realizations of the field on a regular grid with the given values ??on an irregular grid is considered. As an example of this algorithm is a special method of assessing the impact of uncertainty in the initial data, due to the limited and irregular network of meteorological stations, the results of numerical modeling of atmospheric processes on the basis of hydrothermodynamic models is offered.

Algorithms for modeling of piecewise constant non-Gaussian processes are a modification of the method of modeling non-Gaussian stationary processes and homogeneous, isotropic and homogeneous fields on the point flows Palma proposed and studied in [12,24,35]. The specifics of discussed in this chapter is in combine algorithms for discrete models of non-Gaussian random processes with models of stationary point flows. The above algorithms are directly connected with the algorithms of stochastic interpolation of random processes and fields specified in the nodes of a regular (or irregular) of the grid in an arbitrary point of the area. Interpolation mechanisms that preserve the properties of the original discrete processes and fields: the one-dimensional distribution for processes - stationarity, for the fields - homogeneity (or homogeneous, isotropic) are considered. The correlation structure of the examined processes and fields are investigated.

Textbook is intended for students and undergraduates of MMD NSU and corresponds to the course " Numerical modeling of discrete random processes and fields" of master basic educational program in "Numerical statistical modeling. Monte Carlo methods. "

2. METHOD OF CONDITIONAL DISTRIBUTIONS

1.1. Gaussian processes with correlation matrices of the Toeplitz form

The first two chapters are devoted to the problems of numerical simulation of scalar and vector Gaussian random sequences and fields of discrete argument with the correlation matrices belonging to the class of Toeplitz. The corresponding algorithms are based on the method of conditional " functions of distribution " , which was proposed by A.S. Marchenko, V.A.Ogorodnikova [20], and generalized by the author to the multidimensional case in [27]. This method is a modification of a general method for modeling non-Gaussian vectors with a given multi-dimensional density distribution (in the literature [15,20,27,39] method known as conditional distributions) to the case of modeling the normal vectors with a given covariance matrix. The main advantage of the method of " conditional expectations " over other linear methods , is that of all the methods for modelling of Gaussian vectors with the correlation matrices R_n of the Toeplitz type, based on linear transformations of the form
$\vec{\xi}_n = A_n\vec{\varphi}_n$, where $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$, nd $\vec{\varphi}_n = (\varphi_1, \ldots, \varphi_n)^T$ is vector of normal independent variables with zero mean and unit variance , and $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$, A_n is some square matrix such that$A_n A_n^T = R_n$, apparently , only this method allows to construct a recursive algorithm to calculate the matrix and , consequently, greatly increase the dimension of the tasks.

Class processes and fields, the correlation structure that describes the different types of Toeplitz matrices is widde enough.

If the wide-sense stationary random process $\xi(t)$, $-\infty < t < \infty$ is defined on any finite set t_i, $i = 1, \ldots, n$ of values of parameter t?, and $t_{i+1} - t_i = \Delta t = const$, then the correlation matrix $R_n$of the vector random variable

$$(\xi(t_1), \ldots, \xi(t_n))^T = (\xi_1, \ldots, \xi_n)^T = \vec{\xi}_n$$

has the form

$$R_n = \left\| \begin{array}{cccc} 1 & r_1 & r_2 & \cdots r_{n-1} \\ r_1 & 1 & r_1 & \cdots r_{n-2} \\ r_2 & r_1 & 1 & \cdots r_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n-1} & r_{n-2} & r_{n-3} & \cdots 1 \end{array} \right\|, \tag{1.1}$$

ie is Toeplitz one. Similarly, if the sett_i of values of parameter t is a countably infinite set ($i = \ldots -1, 0, 1, \ldots$) and $\Delta t = const$, then the vector random variables $\vec{\xi}_{n'}$ corresponding to any finite subset $t_{i'} i' = 1, \ldots, n'$ has the correlation matrices of the form (1.1).
One of the important class of stochastic processes
$\ldots \xi(t_{-1}), \xi(t_0), \xi(t_1), \ldots = \ldots \xi_{-1}, \xi_0, \xi_1, \ldots,$
are autoregressive moving average processes, and mixed processes autoregressive-moving average [8-17], for which the matrix of the type (1.1) for any finite subset $t_{i'}$ are Toeplitz and have a specific characteristic of each of these models, for example, for the process moving average of these matrices are belt.

Now let $\vec{\xi}(t), \ -\infty < t < \infty$ is wide-sense stationary multidimensional process [5], where$\vec{\xi}(t)$ for any fixed $t = t_i$ is a vector of the form
$\vec{\xi}(t_i) = (\xi_1(t_i), \ldots, \xi_p(t_i))^T = \vec{\xi}_i$.
If this process is defined in any finite set t_i, $i = 1, \ldots, n$ values of the parametert, $\Delta t = const$, then the covariance matrix $R_{(n)}$ of the multidimensional vector random variable
$\vec{\xi}(t_i) = (\xi_1(t_i), \ldots, \xi_p(t_i))^T = \vec{\xi}_i$ has a block-Toeplitz form [18]

$$R_{(n)} = \left\| \begin{array}{cccc} R_0 & R_1 & R_2 & \cdots R_{n-1} \\ R_1^T & R_0 & R_1 & \cdots R_{n-2} \\ R_2^T & R_1^T & R_0 & \cdots R_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ R_{n-1}^T & R_{n-2}^T & R_{n-3}^T & \cdots R_0 \end{array} \right\|, \tag{1.2}$$

where $R_k, \ k = 0, \ldots, n-1$ are the matrix $p \times p$. The real positive definite matrix $R_{(n)}$ is symmetric one, but its blocky elements can be both symmetric and nonsymmetric.

If $M\vec{\xi}_{(n)} = \vec{0}$, then covariances $M\xi_i(t_l)\xi_j(t_l + k\Delta t) = r_{ij}(k)$, $M\xi_j(t_l)\xi_i(t_l + k\Delta t) = r_{ji}(k)$ are elements of the matrices R_k, and in the general case $r_{ij}(k) \neq r_{ji}(k)$.

The components of vector $\vec{\xi}(t)$ may be interpreted as random processes

of different physical nature, the vectors $\vec{\xi}(t_1), \ldots, \vec{\xi}(t_n)$ as the corresponding time series on a finite set of equally spaced time points, and the sequence of elements of matrix $R_{(n)}$, equally spaced in blocks $R_k,\ k = 0, \ldots, n-1$ as a set of auto- and cross-covariance functions for time shifts $k\Delta t$. In solving of practical problems, for example, when modeling the joint time series of various meteorological elements of the matrix R_k is most often nonsymmetrical (only in some rare cases, approximately one admit their symmetry).
If $\vec{\xi}(t)$ for any t is a vector in the Euclidean space, then the corresponding stationary process can be interpreted as a process of flow velocity and wind , and the sequence of blocks R_k of the matrix (1.2) for the vectors $\vec{\xi}(t_1), \ldots, \vec{\xi}(t_n)$ as the cross sections of the covariance tensor

$K_{\vec{\xi}}(t_i, t_j) = M\vec{\xi}(t_i) \times \vec{\xi}(t_j) = K_{\vec{\xi}}(k\Delta t)$,

where $\times$ denotes the tensor product of vectors as well $k\Delta t = t_j - t_i,\ i, j = 1, \ldots, n$. The elements themselves of the covariance tensor depend of the coordinate system of the Euclidean vector , but there is a set of functions from the elements of the covariance tensor (or invariants) that do not depend on the choice of coordinates. Thus the block- Toeplitz matrix (1.2) as a function of the coordinate system of the Euclidean vector $\vec{\xi}(t)$ at a fixed $t = t_i,\quad i = 1, \ldots, n$ is a non-negative definite tensor function of discrete arguments t_i and t_j.

The next class of stochastic processes whose covariance structure is described by matrices of the form (1.2) are the spatio-temporal field on a regular and non-regular spatial grid. Time grid as in all previous cases remains regular ($\Delta t = const$). Let $\xi(\vec{x}, t) = \xi(x, y, z, t)$ is stationary random process depending on the spatial and temporal coordinates (in accordance with the classification of MS Bartlett [5], this process also referred to as multi-dimensional process). On a finite set of parameters $x_l, y_l, z_l,\quad l = 1, \ldots, p$ for a fixed $t = t_i$ values of the process $\vec{\xi}(x, t)$ form a vector $\vec{\xi}_p(t_i) = (\vec{\xi}(\vec{x}_1, t_i), \ldots, \vec{\xi}(\vec{x}_p, t_i))^T$ with a covariance matrix $M\vec{\xi}_p(t_i)\vec{\xi}_p(t_i) = R_0$. The vector $\vec{\xi}_p(t_i)$ is interpreted as a spatial random field on the grid coordinates of nodes $\vec{x}_l,\ l = 1, \ldots, p$.

On a finite set of values t_i, $i = 1, \ldots, n$ of the parameter t, $\Delta t = const$ multi-dimensional vector random variable

$\vec{\xi}(t_i) == (\vec{\xi}_p^T(t_1), \ldots, \vec{\xi}_p^T(t_n))^T = (\vec{\xi}_1^T, \ldots, \vec{\xi}_n^T)^T = \vec{\xi}_{(n)}$

is interpreted as a stationary time spatio-temporal field with covariance matrix $R_{(n)}$ of the form (1.2).
Class of spatial fields, which can be built on the basis of the approaches that are discussed in this tutorial, is determined by the structure of vector $\vec{\xi}_p(t_i)$

and structure of covariance matrix R_0. If the spatial grid$\vec{x}_1, \ldots, \vec{x}_p$ is irregular, then for a fixed t_i field is a vector $\vec{\xi}_p(t_i) = (\xi(\vec{x}_1, t_i), \ldots, \xi(\vec{x}_p, t_i))^T$ with covariance matrix R_0, in general, of any kind. If a spatial grid is regular ,then the components of vector $\vec{\xi}_p(t_i)$ may be combined into a system of sub-vectors , and the matrix R_0 is block one. If the field is homogeneous, its blocks are asymmetric Toeplitz matrices, but if the field is homogeneous isotropic, these blocks are also symmetrical. Therefore, the covariance matrix of the spatio-temporal field in which for each fixed t_i spatial field is homogeneous, is a block-Toeplitz and, thus, its blocks are also Toeplitz.

It should be noted that the method of "conditional expectations", which is the basis of modeling the classes of processes and fields with a discrete argument focuses on the direct use of the covariance matrix of the simulated processes. The corresponding matrices can be obtained by treating the real hydrometeorological information considering specificity of the model used.

Finally, we consider one more class of processes, the covariance structure of which is described by the matrix of type (1.2). This class is periodically nonstationary processes in the broad sense, or, as they are commonly called [6], periodically correlated random processes (PCRP). Let $\xi(t),\ -\infty < t < \infty$ is periodically correlated random process, for which the mean, variance and correlation function are periodic functions

$M\xi(t+T) = M\xi(t)$,
$D\xi(t+T) = D\xi(t)$,
$K(t+T,\tau) = K(t,\tau)$,

where $K(t,\tau) = M(\xi(t) - M\xi(t))(\xi(t+\tau) - M\xi(t+\tau))$. If $\xi(t)$ defined on a finite set $t_i\ i = 1, \ldots, m$ of values of parameter t, $\Delta t = const$, where $m = n \times p$, $p\Delta t = T$, then the covariance matrix $R_{(n)}$ of the vector random variable

$(\xi(t_1), \ldots, \xi(t_m))^T =$
$= (\xi(t_1), \ldots, \xi(t_p), \ldots, \xi(t_{m-p+1}), \ldots, \xi(t_m))^T =$
$= (\vec{\xi}_1^T, \ldots, \vec{\xi}_n^T)^T = \vec{\xi}_{(m)}$

has the form (1.2), in which elements of the blocks $R_k,\ k = 0, \ldots, n-1$, unlike all previous cases, depend on the time variable t_i. The covariance structure of the multi-dimensional and vectorial periodically correlated processes of the discrete argument may be described by matrices of the type (1.2), by analogy with the above cases.

The above cases are not exhaust the class of processes and fields, for which the construction the apparatus can be used Toeplitz matrices. For example, the total covariance matrix of the process may not have the property of block Toeplitz, but each of its block be a Toeplitz matrix. As in

the scalar case, to increase the dimension of the task the method of conditional "expectations" can be combined with multi-dimensional models of autoregression, moving average and mixed models of autoregressive-moving average.

1.2. Method of conditional distribution for simulation of Gaussian vectors with an arbitrary covariance matrix

In the general case, n -dimensional Gaussian vector
$\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$ with an arbitrary covariance matrix $R_n = (r_{ij}), \quad i, j = 1, \ldots, n$

$$R_n = \left\| \begin{array}{llll} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{12} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{array} \right\| \tag{2.1}$$

and zero mean can be obtained by the linear transformation [15]

$$\vec{\xi}_n = A_n \vec{\varphi}_n, \tag{2.2}$$

where $\vec{\varphi}_n = (\varphi_1, \ldots, \varphi_n)^T$ is a Gaussian vector with zero mean and the identity correlation matrix, and A_n is a lower triangular matrix such that$A_n A_n^T = R_n$. The matrix A_n

$$A_n = \left\| \begin{array}{llll} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right\| \tag{2.3}$$

is obtained with well-known recurrent formulas [7,44]

$$\begin{array}{c} a_{11} = r_{11}^{1/2}, \quad a_{i1} = r_{i1}/a_{11}, \\ a_{ij} = \dfrac{r_{ij} - \sum\limits_{k=1}^{j-1} a_{ik} a_{jk}}{a_{ii}}, \\ i = 2, \ldots, n, \;\; j = 2, \ldots, i \end{array} \tag{2.4}$$

However, when the dimension of the vector $\vec\xi$ is large, this method fails because of the memory restrictions and numerical instability problems.

Another approach to solving this problem follows from the general method of modeling of a random vector

$$\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T = (\vec{\xi}_q^T, \vec{\xi}_{n-q}^T)^T,$$
$$\vec{\xi}_q = (\xi_1, \ldots, \xi_q)^T, \quad \vec{\xi}_{n-q} = (\xi_{q=1}, \ldots, \xi_n)^T$$

with probability density

$$f(\vec{x}_q, \vec{x}_{n-q})^T,$$
$$\vec{x}_q = (x_1, \ldots, x_q)^T, \quad \vec{x}_{n-q} = (x_{q=1}, \ldots, x_n)^T,$$

for which there is a representation

$$f(\vec{x}_q, \vec{x}_{n-q}) = f_1(\vec{x}_q) f_2(\vec{x}_{n-q} | \vec{x}_{n-q}),$$

where $f_1(\vec{x}_q)$ is the partial density of the vector $\vec{\xi}_q$, and $f_2(\vec{x}_{n-q}|\vec{x}_{n-q})$ is the conditional density of vector $\vec{\xi}_{n-q}$ subject to $\vec{\xi}_q = \vec{x}_q$. If $f(\vec{x}_q, \vec{x}_{n-q}) = f(\vec{x})$ is a multivariate normal density of the vector $\vec{\xi}_n$

$$f(\vec{x}_n) = \frac{1}{2\pi^{n/2} |R_n|^{1/2}} \exp(-\frac{1}{2}(\vec{x}_n - \vec{\mu}_n)^T R_n^{-1} (\vec{x}_n - \vec{\mu}_n)) \tag{2.5}$$

with mathematical expectation
$\vec{\mu}_n = M\vec{\xi}_n = (\vec{\mu}_q^T, \vec{\mu}_{n-q}^T)^T$
and non-singular covariance matrix

$$M(\vec{\xi}_n - \vec{\mu}_n)(\vec{\xi}_n - \vec{\mu}_n)^T = \left\| \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right\| = R_n,$$

then $f_1(\vec{x}_q)$ has the form (2.5), and the conditional density as is well known [1], is

$$\begin{array}{l} f_2(\vec{x}_{n-q}|\vec{x}_{n-q}) = \frac{1}{2\pi^{(n-q)/2}|R_{22.1}|^{1/2}} \times \\ \times \exp(-\frac{1}{2}(\vec{x}_{n-q} - \vec{\nu}_{n-q}(\vec{x}_q))^T R_{22.1}^{-1} (\vec{x}_{n-q} - \vec{\nu}_{n-q}(\vec{x}_q)). \end{array} . \tag{2.6}$$

Here R_{11} and R_{22} are covariance matrices of vectors $\vec{\xi}_q$ and $\vec{\xi}_{n-q}$, R_{12} and R_{21} are the corresponding cross-covariance matrix,

$$\vec{\nu}_{n-q}(\vec{x}_q) = M(\vec{x}_{n-q}|\vec{x}_q) = \vec{\mu}_{n-q} + R_{21} R_{11}^{-1} (\vec{x}_q - \vec{\mu}_q) \ . \tag{2.7}$$

is a conditional mean, provided $\vec{\xi}_q = \vec{x}_q$, $R_{21}R_{11}^{-1}$ is a matrix of regression coefficients $\vec{\xi}_q$ on $\vec{x}_{n-q}$, and

$$R_{22.1} = R_{22} - R_{21} R_{11}^{-1} R_{12} \tag{2.8}$$

is a matrix of partial covariance coefficients, which is independent of $\vec{x}_{n-q}$. . In this case the algorithm of modeling the vector $\vec{\xi}_n$ reduces to modeling of the vector $\vec{\xi}_q$ with the density (2.5), and vector $\vec{\xi}_{n-q}$ with the density $f_2(\vec{x}_{n-q}|\vec{x}_q)$ of the form (2.7) with the help of the following formula (see [15]).

$$\vec{\xi}_{n-q} = \vec{\mu}_{n-q} + R_{21}R_{11}^{-1}(\vec{x}_q - \vec{\mu}_q) + A_{22.1}\vec{\varphi}_{n-q}, \tag{2.9}$$

where $\vec{\varphi}_{n-q} = (\varphi_{q+1}, \ldots, \varphi_n)^T$ are independent of one another and of $\vec{\xi}_q$ normal random variables with zero mean and unit variance, and a lower triangular matrix $A_{22.1}$ such that $A_{22.1}A_{22.1}^T = R_{22.1}$. Vector

$$\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T = (\vec{\xi}_q^T, \vec{\xi}_{n-q}^T)^T,$$

obviously, has normal distribution with the mean $\vec{\mu}_n = (\vec{\mu}_q^T, \vec{\mu}_{n-q}^T)^T$ mean and covariance matrix (2.1). In what follows without violating generality, we assume $\vec{\mu}_n$ a zero vector.
The procedure for modeling component-vector $\vec{\xi}_n$ follows from the representation of the density

$$f_2(\vec{x}_{n-q}|\vec{x}_q) = f(\vec{x}_1, \ldots, \vec{x}_n)$$

in the form [25,40]

$$f(x_1, \ldots, x_n) = f(x_1)f(x_2|x_1)f(x_3|x_2, x_1)\ldots f(x_n|x_{n-1}, x_{n-2}, \ldots, x_1.)$$

Thus, the model sequence of Gaussian variables $\xi_1, \xi_2, \ldots, \xi_n$ (or vector $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$ with zero mean and the correlation matrix R_n) may be obtained by the following chain of transformations

$$\begin{array}{l} \xi_1 = d_0\varphi_1, \\ \xi_2 = \vec{b}^T[1]J_1\vec{\xi}_1 + d_1\varphi_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_n = \vec{b}^T[n-1]J_{n-1}\vec{\xi}_{n-1} + d_{n-1}\varphi_n, \end{array} \tag{2.10}$$

where, as above, $\varphi_i, \quad i = 1, \ldots, n$ are independent of one another and of ξ_i Gaussian variables with zero mean and unit variance, $b_i[k]$ and d_k are scalar coefficients. Here we use the notation

$$\vec{b}[k] = (b_1[k], \ldots, b_k[k])^T, \quad \vec{\xi}_k = (\xi_1, \ldots, \xi_k)^T,$$

$$J_k = \left\| \begin{array}{cccc} 0 & 0 & \cdots 0 & 1 \\ 0 & 0 & \cdots 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots 0 & 0 \end{array} \right\|, \tag{2.11}$$

where J_k is the permutation square matrix of order k. This method is known (see, for example, [3.4]), as a method of "conditional expectation". Here b[k-1] is a vector of regression ξ_k on $\vec{\xi}_{k-1}$; $b_k[k]$ is the partial correlation between ξ_k and ξ_1 for fixed values of the intermediate scalar variables $\xi_{k-1}, \ldots, \xi_0$; d^2_{k-1} is the average squared difference between ξ_k and $\vec{b}[k-1]J_{k-1}\vec{\xi}_{k-1}$.

Let us consider some properties of the matrix. J_k. It is obviously, that

$$J_k^T = J_k, \quad J_k J_k = I_k, \quad J_k^T = J_k,$$

where I_k is the identity matrix of order k. Let S_k is square matrix is symmetric about the main diagonal. Then

$$J_k S_k J_k = \tilde{S}_k,$$

where $\tilde{S}_k$ is the matrix obtained from S_k by transposition of relatively secondary diagonal. If $S_k = \tilde{S}_k$ is matrix, symmetric with respect to the primary and secondary diagonal simultaneously, then $J_k S_k J_k = S_k$. . In the literature [18], the matrix type (2.11) called reflection matrix. It is also obvious

$$J_k \left\| \begin{array}{c} \xi_1 \\ \vdots \\ \xi_k \end{array} \right\| = \left\| \begin{array}{c} \xi_k \\ \vdots \\ \xi_1 \end{array} \right\|.$$

The vectors b[k-1], k=2,...,n are connected with the given elements of the correlation matrix R by the equation

$$J_{k-1} R_{k-1} J_{k-1} \vec{b}[k-1] = \vec{r}[k-1] \tag{2.12}$$

where the vector $\vec{r}[k-1] = (r_{k-1,k}, \ldots, r_{1k})^T$ is associated with the last column of matrix $R_k$by the relation

$$\vec{r}[k-1] = J_k \left\| \begin{array}{c} r_{1k} \\ \vdots \\ r_{k-1,k} \end{array} \right\| = \left\| \begin{array}{c} r_{k-1,k} \\ \vdots \\ r_{1k} \end{array} \right\|, .$$

and d_{k-1} are given by

$$\begin{array}{l} d_0^2 = r_{11}, \\ d_{k-1}^2 = r_{kk} - \vec{b}^T[k-1]\vec{r}[k-1]. \end{array} \tag{2.13}$$

Let us consider some of the properties of this method. For this, we represent the matrix
$R_k, \quad k = 1, \ldots, n$ in the block form

$$R_k = \left\| \begin{array}{cc} R_{k-1} & J_{k-1}\vec{r}[k-1] \\ \vec{r}^T[k-1]\ J_{k-1} & r_{kk} \end{array} \right\|. \tag{2.14}$$

If R_n is a positive definite matrix, then all its principal minors are positive, so from

$$\det(R_k) = \det(R_{k-1})(r_{kk} - \vec{r}^T[k-1]J_{k-1}R_{k-1}^{-1}J_{k-1}\vec{r}[k-1]) =$$
$$= \det(R_{k-1})(r_{kk} - \vec{r}^T[k-1]\vec{b}[k-1]) = \det(R_{k-1})d_{k-1}^2$$

It follows that $d_k^2 > 0$ for all $k = 1, \ldots, n, \ \ (d_0 = 1)$. We now represent the transformation (2.10) in the form

$$B_n\vec{\xi}_n = D_n\varphi_n, \tag{2.15}$$

where

$$B_n = \left\| \begin{array}{cccccc} 1 & 0 & \cdots & 0 & 0 \\ -b_1[1] & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_1[n-1] & & \cdots & -b_1[n-1] & 1 \end{array} \right\|, \tag{2.16}$$

$$D_n = \left\| \begin{array}{cccc} d_0 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_{n-1} \end{array} \right\|. \tag{2.17}$$

From (2.15) we have

$$\vec{\xi}_n = B_n^{-1}D_n\vec{\varphi}_n = A_n\vec{\varphi}_n.$$

Then

$$R_n = M\vec{\xi}_n\vec{\xi}_n^T = A_nA_n^T = B_n^{-1}D_nD_n^T(B_n^{-1})^T =$$
$$= B_n^{-1}D_nD_n(B_n^T)^{-1}.$$

Since $d_k^2 > 0, \ k = 0, \ldots, n-1$, the matrix A_n is nonsingular, therefore,

$$R_n^{-1} = B_n^TD_n^{-1}D_n^{-1}B_n = T_n^TT_n, \ l \tag{2.18}$$

where $T_n = (t_{ij})$ is the lower triangular matrix. Thus, if R_n - - a positive definite matrix, then the transformation (2.8) is a non-singular linear transformation and the vector $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$ is normally distributed with

mean zero and correlation matrix R_n. The lower triangular matrix B_n is obtained by the decomposition R_n^{-1} on product of the upper (T_n^T) and lower (T_n) triangular matrices,, and then dividing each j-th row of the matrix T_n to its diagonal element belonging to the same line, and D_n is the inverse of a diagonal matrix, formed from the diagonal elements of the matrix T_n.

Indeed, if $R_n$is positive definite and, R_n^{-1} is known then $t_{ii} \neq 0$, $i = 1, \ldots, n$ and

$$T_n = \left\| \begin{array}{cccc} t_{11} & 0 & \cdots & 0 \\ t_{21} & t_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{array} \right\| =$$

$$= \left\| \begin{array}{cccc} t_{11} & 0 & \cdots & 0 \\ 0 & t_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t_{nn} \end{array} \right\| = \left\| \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ t_{21}/t_{22} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ t_{21}/t_{22} & t_{21}/t_{22} & \cdots & 1 \end{array} \right\|.$$

It is also obvious that the matrix A_n is associated with T_n by the ratio $T_n = A_n^{-1}$.

Realization of the algorithm (2.10) for any correlation matrix R_n is associated with the solution of equations (2.12), or with a triangular decomposition (2.18) of this matrix. Therefore, the algorithm by a labor is comparable with the algorithm (2.4).

The problem is greatly simplified if we are talking about the simulation of stationary sequences$(\xi_1, \ldots, \xi_n)$ with the joint correlation matrix of the Toeplitz (or stationary) form. In this case, the method of conditional expectations has the fundamental advantage over the algorithm (2.4), and allows us to calculate the matrix elements by a simple recursive formulas.

1.3. Simulation of the Gaussian vectors with correlation matrix of stationary type

The problem consists in construction of the Gaussian vector $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$ with zero mean and the correlation matrix R_n of the form

$$R_n = \left\| \begin{array}{cccc} 1 & r_1 & \cdots & r_{n-1} \\ r_1 & 1 & \cdots & r_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n-1} & r_{n-2} & \cdots & 1 \end{array} \right\| \tag{3.1}$$

Let

$$R_k\vec{b}[k] = \vec{r}_k, \quad k = 1, 2, \ldots, n-1, \tag{3.2}$$

$$R_k = \left\| \begin{array}{cccc} 1 & r_1 & \cdots & r_{k-1} \\ r_1 & 1 & \cdots & r_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ r_{k-1} & r_{k-2} & \cdots & 1 \end{array} \right\|$$

$$R_{k+1} = \left\| \begin{array}{cc} R_k & J_k\vec{r}_k \\ \vec{r}^T k J_k & r_1 \end{array} \right\|, \tag{3.3}$$

$\vec{r}_k = (r_1, \ldots, r_k)^T, \quad \vec{r}_{k+1} = (\vec{r}_k^T, r_{k+1})^T,$
$\vec{b}[k] = (b_1[k], \ldots, b_k[k])^T, \quad \vec{b}[k+1] = (b_1[k+1], \ldots, b_{k+1}[k+1])^T,$

Using this notation, we rewrite the above considered algorithm **??**

$$\begin{array}{l} \xi_1 = \varphi_1 \\ \xi_2 = \vec{b}^T[1]J_1\vec{\xi}_1 + d_1\varphi_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_n = \vec{b}^T[n-1]J_{n-1}\vec{\xi}_{n-1} + d_{n-1}\varphi_n, \end{array} \tag{3.4}$$

where $\varphi_1, \varphi_2, .., \varphi_n$ are independent Gaussian variables with zero means and unit variances,

$$\vec{\xi}_k = (\xi_1, \ldots, \xi_k)^T, \quad k = 1, \ldots, n-1.$$

Let us consider the inverse block matrix for R_{k+1}

$$R_{k+1}^{-1} = \left\| \begin{array}{cc} R_k^{-1} + J_k\vec{b}[k]\vec{b}^T[k]J_k/d_k^2 & \cdots - J_k\vec{b}[k]/d_k^2 \\ -\vec{b}^T[k]J_k/d_k^2 & \cdots\cdots\vdots 1/d_k^2 \end{array} \right\|, \tag{3.5}$$

where

$$d_k^2 = 1 - \vec{r}_k^T R_k\vec{r}_k = 1 - \vec{r}_k^T\vec{b}[k].$$

Substitution of matrix (5) into the equality

$$\vec{b}[k+1] = R_{k+1}^{-1}\vec{r}_{k+1}$$

to solving equation

$$R_{k+1}\vec{b}[k+1] = \vec{r}_{k+1}$$

for $\vec{b}[k+1]$ gives [14]

$$\begin{aligned}
&(b_1[k+1],\ldots,b_k[k+1])^T = \vec{b}[k] - b_{k+1}[k+1]J_k\vec{b}[k],\\
&b_{k+1}[k+1] = (r_{k+1} - \vec{r}_k^T J_k \vec{b}[k])/d_k^2,\\
&d_k^2 = 1 - \vec{r}_k^T \vec{b}[k]
\end{aligned} \tag{3.6}$$

The computation begins with $k=1$ with the original value $\vec{b}[1] = b_1[1] = r_1$ and at $k = n-1$ gives the solution $\vec{b}[n]$ of the equation $R_n\vec{b}[n] = r_n$. Relations (3.4) explain the probabilistic sense of the parameters $\vec{b}[k]$, $b_{k+1}[k+1]$ and d_k^2; namely, $\vec{b}[k]$ is the regression vector for ξ_{k+1} on $\vec{\xi}_k$, $\beta_{k+1}[k+1]$ is the partial correlation coefficient between ξ_{k+1} and ξ_1 provided the scalars $\xi_k,\ldots,\xi_2$ are fixed; d_k^2 is the mean square difference between ξ_{k+1} and $\vec{b}^T[k]J_k\vec{\xi}_k$.

This algorithm calculates $\vec{b}[k]$ and d_k for arbitrary matrix of the type of (3.1). Note that when the values of d_k and $\vec{b}[k]$ are calculated, only d_{k-1} and the vector $\vec{b}[k-1]$ consisting of $k-1$ components must be stored. This algorithm is simple in program realization, and it is often used to simulate the Gaussian vectors of large dimension [91].

For the positive definite Toeplitz correlation matrix R_n, as in the general case, the following relation is valid

$$\det(R_{k+1}) = \det(R_k)d_k^2 > 0$$

for all $k = 1,\ldots,n-1$. Recurrent calculation gives

$$\det(R_n) = \det(R_{n-1})d_{n-1}^2 = \det(R_{n-2})d_{n-2}^2 d_{n-1}^2 = \ldots$$

thus

$$\det(R_n) = d_1^2 d_2^2 \cdots d_{n-1}^2, \quad d_1^2 = 1.$$

Obviously, the inverse statement is also valid: if $d_1^2 > 0,\ldots,d_{n-1}^2 > 0$, then R_n is positive definite. To illustrate it, let us represent scheme (3.4) in the form

$$B_n\vec{\xi}_n = D_n\vec{\varphi}_n, \tag{3.7}$$

where

$$B_n = \left\| \begin{array}{cccccc} 1 & 0 & \cdots & 0 & 0 \\ -b_1[1] & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_1[n-1] & & \cdots & -b_1[n-1] & 1 \end{array} \right\|, \tag{3.8}$$

$$D_n = \left\| \begin{array}{cccc} d_0 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_{n-1} \end{array} \right\|. \tag{3.9}$$

is a positive definite matrix because $d_1 > 0, \ldots, d_{n-1} > 0$. Then it follows from

$$R_n = B_n^{-1} D_n D_n (B_n^T)^{-1}$$

that R_n is a positive definite matrix. This fact can be made use of to numerically test whether a stationary matrix is positive definite.

Using (3.6) we can express $\vec{b}[k]$ is terms of $\vec{b}[k-1]$ and then it yields

$$d_k^2 = (1 - b_k^2[k]) d_{k-1}^2. \tag{3.10}$$

Thus, $b_k^2[k] < 1$ for all $k = 1, \ldots, n-1$ ($d_0^2 = 1$) and, also, [91]

$$1 = d_0^2 \geq d_1^2 \geq \ldots \geq d_{n-1}^2.$$

1.4. Regularization of the algorithm

Algorithm (3.6) proves to be numerically unstable [20,27] for some correlation matrices. Indeed, it may appear that errors in r_k will result in errors of calculation of the correlation coefficient $b_{k+1}[k+1]$, so that it becomes equal to or larger than unit, thus resulting in the incorrect inequality $d_k^2 \leq 0$. For example, in the case when

$$r_h = \exp(-\alpha h^2), \quad \alpha = 10^{-5},$$

the numerical instability arises even after the first several steps.

It should be noted that the correlation function $r_n = \exp(-\alpha h^2)$ for small α, often used in practice, is one of the most "difficult correlation functions" in the sense that algorithm (3.6) leads to unstatble calculations at the first several steps. Regularization methods of the algorithms of type (3.6) are described in [20,27].

Below we consider some simple regularization methods. It should be noted that due to restricted possibilities of modern computers it is impossible to construct an algorithm that is absolutely stable for any correlation function. In some cases, we have to construct a time series with a correlation function that is an approximation of the desired correlation function r_k (see, e.g., [8]).

1.4.1. Simulation of stationary vectors with correlation matrix $G_n = (1-\varepsilon)R_n + \varepsilon I_n$

In simulating time series, the correlation function r_k is usually defined by an analytical expression of positive defined functions. However, the values of r_k are always calculated approximately with an error Δ_k, so that $\hat{r}_k = r_k + \Delta_k$. The matrix $\hat{R}_n = (\hat{r}_{|i-j|})$ may not be positive definite, especially if the eigenvalue of the exact matrix $R_n = (r_{|i-j|})$ is close to zero.

Let ε be an arbitrary number from the interval [0, 1], and let

$$g_0 = 1, \quad g_n = (1-\varepsilon)r_n, \quad h = 1, 2, \dots \ l \tag{4.1}$$

Consider the following stationary (Toeplitz) $n \times n$ matrix:

$$G_n = (g_{|i-j|}) = \varepsilon I_n + (1-\varepsilon)R_n, \ l \tag{4.2}$$

where I_n is the identity matrix.

In this case, instead of the vector $\vec{\xi} = (\xi_1, \dots, \xi_n)^T$, we simulate a Gaussian vector $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)^T$ with a stationary correlation matrix G_n. Note that choosing ε, it is possible to make the matrix G_n positive definite and carry out stable calculations in algorithm (3.6). The parameter ε is experimentally found, using the Durbin algorithm, starting, for example, with $\varepsilon = 10^{-10}$. As experiments show, the stability is often attained even for small values of ε. For example, for $r_h = \exp(-\alpha h^2)$, $\alpha = 10^{-5}$, $E = 10^{-5}$, algorithm (3.6) gives stable results at steps numbered by thousands.

If high accuracy is not required, it is possible to use the sequence $\zeta_1, \dots, \zeta_n$, since

$$|g_k - r_k| = |(1-\varepsilon)r_k - r_k| = \varepsilon|r_k| < \varepsilon, \quad k \geq 1.$$

Note that the following relation is valid:

$$\zeta_k = \sqrt{1-\varepsilon}\,\xi_k + \sqrt{\varepsilon}\,\eta_k,$$

where $\eta_1, \eta_2, \dots, \eta_n$ are mutually independent (and independent of $\xi_1, \xi_2, \dots, \xi_n$) random variables with zero means and unit variances.

1.4.2. Doubled algorithm

The above-described stable algorithm for the matrix G_n can be used to improve calculation of the partial correlation coefficients $b_{k+1}[k+1]$,

$k = 1, \ldots, n-2$ [91]. The quantity $1 - \vec{r}_k^T \vec{b}[k] = d_k^2$ in the denominator of the second expression in (3.6) is often very small even for small values of k. This means that $\vec{r}_k^T \vec{b}[k]$ is close to unit. As the absolute value of $b_{k+1}[k+1]$ is not larger than unit, the difference in the numerator of (3.6) is very small. All these lead to non-stability of algorithm (3.6). To improve the stability, it is necessary to change the method of calculation of the partial correlation coefficients.

Assume for a while that $r_{k+1} \neq 0$, and rewrite $b_{k+1}[k+1]$ in the form

$$b_{k+1}[k+1] = r_{k+1} \det(V_k) / \det(W_k), \ l \tag{4.3}$$

where V_k and W_k are matrices

$$V_k = I_k - J_k \vec{b}[k] \vec{r}_k^T / r_{k+1}, \quad W_k = I_k - \vec{b}[k] \vec{r}_k^T.$$

The determinants in (4.3) are usually very small; therefore, we try to perform equivalent transformations to avoid the division in small values. The matrix $G_n = \varepsilon I_n + (1-\varepsilon) R_n$ is close to R_n for small ε, hence, the matrices

$$\tilde{V}_k = I_k - J_k \vec{\beta}[k] / \vec{g}_{k+1}^T, \quad \tilde{W}_k = I_k - \vec{\beta}[k] \vec{g}_k^T$$

are close to V_k and W_k, respectively. Here $\vec{\beta}[k]$ is the regression vector from (3.6) for $G_n = (g_{|i-j|})$. The smaller ε, the closer the matrices $V_k \tilde{V}_k$ and $W_k \tilde{W}_k$ to the identity matrix. Instead of (4.3), we can write

$$b_{k+1}[k+1] = \frac{r_{k+1} \det(\tilde{V}_k) \det(V_k \tilde{V}_k^{-1})}{\det(\tilde{W}_k) \det(W_k \tilde{W}_k^{-1})} = \frac{\beta_{k+1}[k+1]}{1-\varepsilon} \frac{\det(V_k \tilde{V}_k^{-1})}{\det(W_k \tilde{W}_k^{-1})}.$$

Note that the choice of ε leads to the stable calculation of $\beta_{k+1}[k+1]$. Next, simple manipulations give:

$$b_{k+1}[k+1] = \frac{\beta_{k+1}[k+1] / (1-\varepsilon) + \vec{r}_k^T J_k (\vec{\beta}[k] - \vec{b}[k]) / \delta_k^2}{r_k^T \left[(1-\varepsilon) \vec{\beta}[k] \right] / \delta_k^2 + 1}. \ l \tag{4.4}$$

This formula is also valid if $r_{k+1} = 0$. In (4.4), δ_k^2 is a residual variance from (3.6) for G_n.

Experiments show that the doubled algorithm (3.6) must be used together with the regularization technique described above. This approach strongly improves the numerical stability. This practically means that instead of r_k the slightly changed function is used: $r_k^0 = 1$ for $k = 0$ and $r_k^0 = (1-\varepsilon) r_k$ for $k \geq 1$, where ε is of order 10^{-9}.

1.4.3. Control of calculation accuracy

As algorithm (3.6) is nonlinear, the total investigation of calculation accuracy for a large n is rather tedious. In this case, since construction of a time series with some given correlation function is the finite objective of calculations, deviation of correlations between elements of the constructed sequence $\xi_1, \xi_2, \ldots, \xi_n$ from the given values $r_{|i-j|}$ being of immediate interest. If the matrices B_n and D_n are calculated with some errors such that results of calculations are true in the probability sense (i.e., all $\delta_k^2 > 0$), then the vector $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T$ has the correlation matrix $\hat{R}_n = B_n^{-1} D_n D_n (B_n^T)^{-1}$ which is not only distinguished from R_n, but is, generally speaking, a non-Toeplitz matrix.

The largest by absolute value component δ of the residual vector $R_n \vec{b}[n] - \vec{r}_n$ gives a rough picture of the distinction between the quantities $\hat{r}_{ij}$ and $r_{|i-j|}$. Experiments have shown that the difference $|\hat{r}_{ij} - r_{|i-j|}|$ is larger than δ by the factor of 10^2.

Comparison of elements of the first columns of the matrices $\hat{R}_n$ and R_n is more reliable. The first column $\vec{x}$ of the matrix $\hat{R}_n$ is equal to

$$\vec{x} = B_n^{-1} D_n D_n B_n^{T-1} \vec{e}_1,$$

where $\vec{e}_1 = (1, 0, \ldots, 0)^T$ is n-dimensional vector. It is easy to see that $\vec{x} = B_n^{-1} \vec{e}_1$ or $B_n \vec{x} - \vec{e}_1$ follows from $D_n D_n B_n^{T-1} \vec{e}_1 = B_n \vec{x}$. However B_n is the lower triangular matrix, therefore components of the vector $\vec{x}$ are recurrently calculated in the process of calculations as the next row of the matrix B_n is obtained.

Comparison of all the elements of the matrices $\hat{R}_n$ and R_n gives a complete information about calculation accuracy, but this work calls for tedious calculations and it cannot be realized for a routine monitoring of the calculation accuracy. For example, elements $\hat{r}_{ij}$ and $r_{|i-j|}$ of the matrices $\hat{R}_n$ and R_n are compared in Tables 1. and 2. Here $n = 1000,\ i, j = 1, 500, 1000$. The data represented in Table 1. correspond to algorithm (3.6). Two types of correlation functions are considered:

$$r_{|i-j|} = \begin{cases} 1 & |i-j| = 0 \\ (1-\varepsilon)\ \exp\left(-\alpha |i-j|^2\right), & |i-j| \neq 0 \end{cases}, \tag{4.5}$$

in Table 1., and

$$r_{|i-j|} = \exp\left(a|i-j|^\alpha / (1 + b|i-j|^\beta)\right) \tag{4.6}$$

in Table 1.2, where

$$a = 10^{-2}, \quad \alpha = 1.4, \quad b = 0.1, \quad \beta = 0.5.$$

(The latter functions are peculiar to time series of the air temperature.)

Table 1: Comparison of elements of the matrices R_n and $\hat{R}_n$ for correlations (4.5)

i\j	1	500	1000
1	1.000000000 10^0 1.000000000 10^0	0.8208417778 10^{-3} 0.8208417776 10^{-3}	0.4539947577 10^{-6} 0.4539934909 10^{-6}
500		1.000000000 10^0 0.9999998520 10^0	0.8208417778 10^{-3} 0.8208417089 10^{-3}
1000			1.000000000 10^0 0.9998892260 10^0

In every square of each table the upper number is an element of the exact matrix R_n, and the lower one is a corresponding element of the matrix, resulted from calculations. One can see that calculation errors tend to increase towards the right lower angle of the matrix which becomes non-Toeplitz.

Table 2: Comparison of elements of the matrices R_n and $\hat{R}_n$ for correlations (4.6)

i\j	1	500	1000
1	1.0000000000 10^0 1.0000000000 10^0	0.2132937465 10^{-2} 0.2132937465 10^{-2}	0.9048866059 10^{-3} 0.9048866059 10^{-3}
500		1.0000000000 10^0 0.999999998 10^0	0.2132937465 10^{-2} 0.2132937461 10^{-2}
1000			1.0000000000 10^0 0.9999999710 10^0

If we set $\alpha = 10^{-5}$ and $\varepsilon = 10^{-9}$ in formula (5), then algorithm (6) may be realized only for 100 steps. However if in this case one applies a

doubled algorithm, then calculations may be extended to thousands steps without considerable storing errors. The results are represented in Table 3. Here the upper numbers correspond to the given matrix R_n.

Table 3: Comparison of the first row of the matrices R_n and $\hat{R}_n$ for correlations (4.5) (doubled algorithm)

j	1	500	1000
r_{ij}	$1.000000000\ 10^{0}$	$0.8208417778\ 10^{-3}$	$0.4539947557\ 10^{-5}$
$\hat{r}_{ij}$	$1.000000000\ 10^{0}$	$0.8208417778\ 10^{-3}$	$0.4539947551\ 10^{-5}$

1.5. Simulation of autoregressive processes with a desired correlation structure

In many applications, when large sequences $\xi_1, \xi_2, \ldots, \xi_n$ are simulated, it is necessary to preserve the correlations between ξ_t and ξ_{t+k} only for $k \leq m$, where m is a characteristic correlation scale. This simplifies procedure (2.4) by setting $b_{k+1}[k+1] = 0$ for $k = m$. We will show below that the simulation process transforms to an autoregressive process of finite order. The stationary m-order autoregressive process is defined (see, e.g., [2, 4,22,33]) as a sequence of random variables $\ldots \xi_{-1}, \xi_0, \xi_1, \ldots$ such that

$$\xi_t = \sum_{k=1}^{m} b_k \xi_{t-k} + d\varphi_t, \quad t = \ldots -1, 0, 1, \ldots, \ l \tag{5.1}$$

where $b_1, \ldots, b_m$ are real coefficients (the regression coefficients), d is a constant, normalizing variance of the process, $\ldots, \varphi_{-1}, \varphi_0, \varphi_1, \ldots$ are non-correlated random variables with zero means and unit variances. It is here assumed that the absolute values of the roots of the characteristic equation

$$\lambda^m = \sum_{k=1}^{m} b_k \lambda^{m-k} \ \ l \tag{5.2}$$

are less than unit. In this case the stationary solution of equation (5.1) can be expressed in the form of infinite series (convergent in the mean square sense):

$$\xi_t = \sum_{k=0}^{\infty} c_k \varphi_{t-k}, \ l \tag{5.3}$$

where c_k are appropriate coefficients [3].

Expression (5.3) is often called the infinity backward - directed moving average process. The constant d is convenient to be found from the condition $D\xi_t = 1$. As according to (5.3) φ_t and $\xi_{t-1}, \xi_{t-2}, \ldots$ are non-correlated, we have to set

$$d^2 = 1 - \sum_{h=1}^{m} b_h \rho_h, \ l \tag{5.4}$$

where $\rho_1, \rho_2, \ldots, \rho_m$ are values of the correlation function ρ_h of autoregressive process (5.1). It is not difficult to show that the right-hand side of equation (5.4) is non-negative. Using (5.3) we can define the following difference equation for the correlation function (the Jule - Walker equation, see [2,4])

$$\rho_h = \sum_{k=1}^{m} b_k \rho_{h-k}, \quad h = 1, 2, \ldots \ . \ l \tag{5.5}$$

This equation is solved under the conditions $\rho_0 = 1$, $\rho_{-1} = \rho_1, \ldots, \rho_{-m+1} = \rho_{m-1}$. These conditions normalize the correlation function and ensure its evenness. When the solution is obtained, one defines $\rho_{-N} = \rho_N$ for $N > m$.

The correlation function is uniquely defined by equation (5.5) for arbitrary regression coefficients provided that all the roots of equation (5.2) lie inside the unit circle of the complex plane.

Now, we consider an inverse problem to find out whether it is possible to construct the m-th-order autoregressive process such that its correlation function ρ_h coincide with a given correlation function r_h at initial points. We show that this problem has a unique solution [22].

Thus, keeping in mind that the correlation function is even, we obtain from (5.5):

$$\begin{array}{l} b_1\rho_0 + \cdots + b_m\rho_{m-1} = \rho_1, \\ b_1\rho_1 + \cdots + b_m\rho_{m-2} = \rho_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_1\rho_{m-1} + \cdots + b_m\rho_0 = \rho_m, \end{array} \tag{5.6}$$

where $\rho_0 = 1$. Since ρ_n are uniquely defined by the parameters $b_1, \ldots, b_m$, in (5.6) we have identities.

Now, in equation (5.6), we replace $\rho_0, \ldots, \rho_m$ with the initial values $r_0 = 1, r_1, \ldots, r_m$ of the given correlation function, and consider the obtained relations as a system of equations for $\vec{b}[m] = (b_1, \ldots, b_m)^T$

$$R_m\vec{b}[m] = \vec{r}_m,\ l \tag{5.7}$$

where $\vec{r}_m = (r_1, .., r_m)^T$,

$$R_m = \left\| \begin{array}{lll} 1 & r_1 & \cdots r_{m-1} \\ r_1 & 1 & \cdots r_{m-2} \\ \cdots\cdots\cdots\cdots\cdots & & \\ r_{m-1} & r_{m-2} & \cdots 1 \end{array} \right\|$$

Equation (5.7) is uniquely solvable as R_m is a positive definite matrix (because $1, r_1, \ldots, r_m$ are initial values of a positive definite correlation function r_h). We show that thus determined parameters (provided the matrix R_{m+1} is positive definite) can be considered as coefficients of an autoregressive difference equation of type (5.1), in other words, the corresponding characteristic equation (5.2) has only roots lying inside the unit circle of the complex plane.

Since the autoregressive equation has only one stationary solution of type (5.3) and only one correlation function satisfying equation (5.5), then

the initial values of this correlation function coincide with $1, r_1, \ldots, r_m$ due to equation (5.7). All consequent values are obtained using (5.5).

The process described by equation (5.1) will transfer a stationary state after rather a large number of steps (beginning with arbitrary initial values of ξ_t) due to the properties of the roots of the characteristic equation. However, this method of numerical construction of the realization can be improved. Along with the solution of equation (5.7), it is useful to construct special initial values such that the process have the desired correlation properties (without stationarization procedure). As initial values, we take random variables $\xi_1, \xi_2, \ldots, \xi_m$, stationarily connected, with the correlation coefficients r_h, $h = 1, \ldots, m-1$. We will show that these variables could be calculated by scheme (3.4).

Lemma 1. Assume that a real symmetric positive definite matrix A and a real matrix B of the same dimension are such that the matrix

$$C = A - BAB^T$$

is non-negative definite. Then the absolute values of the real roots of the characteristic equation of C are not larger than 1, and the complex roots lie inside the unite circle of the complex plane.

Proof Let $S = A^{-1/2}BA^{1/2}$. Then

$$C = A^{1/2}(I - SS^T)A^{1/2},$$

so that the matrices $Q = I - SS^T$ and SS^T are non-negative definite. If $\lambda(Q)$ and $\lambda(SS^T)$ are their eigenvalues, then from $\lambda(Q) = 1 - \lambda(SS^T) \geq 0$ we obtain $0 \leq \lambda(SS^T) \leq 1$. In particular,

$$0 \leq \max \lambda(SS^T) \leq 1. \ l \tag{5.8}$$

Obviously,

$$\lambda(S) = \lambda(S^T) = \lambda(B) = \lambda(B^T).$$

Assume, that $\lambda(B)$ is a real eigenvalue of the matrix B, and $\vec{\varphi}$ is the corresponding real eigenvector of the matrix S^T. Then from

$$S^T\vec{\varphi} = \lambda(B)\vec{\varphi}$$

and from inequality (5.8) we obtain

$$\lambda^2(B) = \frac{\vec{\varphi}^T SS^T\vec{\varphi}}{\vec{\varphi}^T\vec{\varphi}} \leq \max \ \lambda(SS^T) \leq 1. \ l \tag{5.9}$$

Assume now that $\lambda(B) = u + iv$ is a complex eigenvalue of the matrix B, and $\vec{\varphi} = \vec{a} + i\vec{b}$ is the corresponding complex eigenvector of the matrix S^T. Since the matrix S^T is real, there exist also a complex conjugate eigenvalue $\lambda^*(B) = u - iv$ and eigenvector $\vec{\varphi}^* = \vec{a} - i\vec{b}$. From

$$S^T\vec{\varphi} = \lambda(B)\vec{\varphi}$$

and

$$S^T\vec{\varphi}^* = \lambda^*(B)\vec{\varphi}^*$$

it follows that

$$\lambda(B)\lambda^*(B)=|\lambda(B)|^2 = \frac{\vec{\varphi}^T SS^T\vec{\varphi}^*}{\vec{\varphi}^T\vec{\varphi}^*} = \frac{\vec{a}^T SS^T\vec{a} + \vec{b}^T SS^T\vec{b}}{\vec{a}^T\vec{a} + \vec{b}^T\vec{b}} \leq \max\ \lambda(SS^T).$$

The equality here arises if only $\vec{a} = a\vec{\sigma},\ \vec{b} = b\vec{\sigma}$, where $\vec{\sigma}$ is a real eigenvector of the matrix SS^T corresponding to the maximal eigenvalue, a and b are arbitrary non-zero real constants.

We show that these conditions cannot be satisfied if $v \neq 0$. We have

$$S^T\vec{a} = u\vec{a} - v\vec{b}, \quad S^T\vec{b} = v\vec{a} + u\vec{b}.$$

If $\vec{a} = a\vec{\sigma}$ and $\vec{b} = b\vec{\sigma}$, then these equalities take the form

$$S^T\vec{\sigma} = (u - vc)\vec{\sigma} = (u + v/c)\vec{\sigma}, \quad c = b/a.$$

Consequently, $u - cv = u + v/c$, or $c + 1/c = 1$, since $v \neq 0$. Hence, c is an imaginary number, and this contradicts to the fact that a and b are real. Thus, for complex eigenvalues we have

$$|\lambda(B)|^2 < \max\ \lambda(SS^T) \leq 1.\ l \tag{5.10}$$

This completes the proof of the statement.

Lemma 2. If R_{m+1} is a positive definite matrix, and $\vec{b}$ is the solution of the equation $R_m\vec{b} = \vec{r}_m$, then the equation

$$\lambda^m = b_1\lambda^{m-1} + \ldots + b_m$$

has no solution equal to $+1$.

Proof Let

$$s_k = b_1[k] + \ldots + b_k[k], \quad s_{k+1} = b_1[k+1] + \ldots + b_{k+1}[k+1],$$

where $(b_1[k], \ldots, b_k[k])^T$ and $(b_1[k+1], \ldots, b_{k+1}[k+1])^T$ are components of the regression vectors $\vec{b}[k]$ and $\vec{b}[k+1]$ from (3.4). Using (3.6) we obtain

$$s_{k+1} = (1 - b_{k+1}[k+1])s_k + b_{k+1}[k+1]. \tag{5.11}$$

Thus, s_k satisfies the first order difference equation with the initial values $s_1 = b_1[1] = r_1$.

The matrix R_{m+1} is positive definite, so the submatrices R_k, $k = 1, \ldots, m$ are also positive definite. Therefore,

$$\det(R_k)(1 - b_k^2[k])d_{k-1}^2 > 0.$$

Consequently, $b_k^2[k] < 1$ for all $k = 1, \ldots, m$. If $s_k \neq 1$, then $s_{k+1} \neq 1$ (otherwise, in view of (11), we would obtain $b_{k+1}[k+1] = 1$). By induction, from $s_1 = r_1 \neq 1$ it follows that

$$s_m = b_1[m] + \ldots + b_m[m] \neq 1.$$

Hence, it also means that $b_1 + \ldots + b_m \neq 1$ because $\vec{b}[m] = \vec{b}$. Thus, the equation

$$\lambda^m = b_1\lambda^{m-1} + \ldots + b_m$$

has no solution equal to $+1$. The following statement is a consequence from Lemma 2.

lemma 3 If R_{m+1} is a positive definite matrix, and $\vec{b}$ is the solution of the equation $R_m\vec{b} = \vec{r}_m$, then the equation

$$\lambda^m = b_1\lambda^{m-1} + \ldots + b_m$$

has no solution equal to -1.

Proof Consider a diagonal matrix Ω_m with elements $\omega_k = (-1)^k$, $k = 1, \ldots, m$, and an extended diagonal matrix Ω_{m+1} arranged in a similar manner. Since $\det(\Omega_{m+1}) \neq 0$, the symmetric matrix

$$R_{m+1}^0 = \Omega_{m+1}R_{m+1}\Omega_{m+1}$$

is positive definite. Moreover, R_{m+1}^0 is Toeplitz matrix since its diagonal elements are equal to unit, the first subdiagonal consists of $(-1)r_1$, the elements of the second subdiagonal are equal to $(-1)^2r_2$, etc.

The first row of this matrix consists of the elements $r_0^0 = 1$, $r_k^0 = (-1)^k r_k = \omega_k r_k$. Thus, R_{m+1}^0 can be represented in the form

$$R_{m+1} = \left\| \begin{array}{cc} R_m^0 & J_m \vec{r}_m^0 \\ \vec{r}_m^{0T} & 1 \end{array} \right\|$$

Application of Lemma 2 to this matrix yields $b_1^0 + \ldots + b_m^0 \neq 1$, where

$$(b_1^0, \ldots, b_m^0)^T = \vec{b}^0 = R_m^{0-1} \vec{r}_m^0 = \\ (\Omega_m R_m \Omega_m)^{-1} \Omega_m \vec{r}_m = \\ = \Omega_m R_m^{-1} \vec{r}_m = \Omega_m \vec{b},$$

so that

$$\omega_1 b_1 + \ldots + \omega_m b_m \neq 1.$$

The last relation proves the statement since it means that

$$(-1)^m \neq b_1 (-1)^{m-1} + \ldots + b_m,$$

because $\omega_k = (-1)^k$.

Now, the main result can be proved.

Theorem 1 If the matrix

$$R_{m+1} = \left\| \begin{array}{cc} R_m & J_m \vec{r}_m \\ \vec{r}_m^T & 1 \end{array} \right\|$$

is positive definite, and $\vec{b}$ is a solution of the equation $R_m \vec{b} = \vec{r}_m$, then the roots of the algebraic equation $\lambda^m = b_1 \lambda^{m-1} + \ldots + b_m$ lie in the unit circle of the complex plane.

Proof The roots of the algebraic equation$\lambda^m = b_1 \lambda^{m-1} + \ldots + b_m$ coincide with the eigenvalues of the matrix

$$\Phi_m = \left\| \begin{array}{ccccc} b_1 & b_2 & \cdots & b_{m-1} & b_m \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{array} \right\|$$

$$= A - \Phi_m A \Phi_m^T = \left\| \begin{array}{ccc} d^2 & 0 & \cdots 0 \\ 0 & 0 & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 0 \end{array} \right\| \qquad (5.12)$$

since det $(R_{m+1}) =$ det $(R_m) d_m^2 > 0$. Thus, the matrix C is non-negative definite, and by Lemma 1, the absolute values of the eigenvalues of this

matrix are not larger than unit. By Lemmas 2, 3, they are not equal to 1 or -1.

Thus, setting in (2.4) $\vec{b}^T[k] = (\vec{b}[m], 0, \ldots, 0)$, $\delta_k = \delta_m$ for all $k = m+1, m+2, \ldots$ we convert the simulated process $\xi_1, \xi_2, \ldots$ to the m-order autoregressive process.

The other ways of proving these statements and the corresponding references are presented in [18].

1.6. The method of conditional expectations for simulation of the conditional realizations of stationary Gaussian sequences.

The method of conditional distributions, as well as the considered variant of this method for the Gaussian case - the method of "conditional expectations" allow to simulate the important for meteorological applications class of random vectors, any set of components which take a fixed value, while the other components have the corresponding conditional distribution. Algorithms from the previous sections of this paper that are based on the method of conditional expectations, allow us to automatically build the realizations of stationary Gaussian processes, provided that at the starting $k$points the value of the process are fixed.

If the values of process are fixed at any set of points that are not initial, then the algorithms of simulation of conditional realizations require the corresponding modification. Some of such methods for conditional simulation of the random Gaussian processes and fields where examined in [27,32].

In this and the following sections we will consider some recursive modeling techniques conditional realizations of stationary processes and homogeneous random fields using of the algorithms discussed in the preceding paragraphs.

Let $\xi_t, \quad t = 1, 1+1/2, 2, 2+1/2, \ldots, n$ is stationary Gaussian sequence of limited length or vector

$$\vec{\xi} = (\xi_1, \xi_{1+1/2}, \ldots, \xi_{(n-1)+1/2}, \xi_n)^T \tag{6.1}$$

with zero mean and a Toeplitz correlation matrix of the form

$$R = \left\| \begin{array}{cccc} 1 & r_{1/2} & r_1 & r_{1+1/2} \ldots r_{n-1} \\ r_{1/2} & 1 & r_{1/2}\, r_1 & \ldots r_{n-1-1/2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ r_{n-1}\, r_{n-1-1/2} & & \ldots & 1 \end{array} \right\|, \tag{6.2}$$

The problem of interpolation of some sequence ξ_t in points

$$t = 1 + 1/2{,}2 + 1/2, \ldots, (n-1) + 1/2$$

provided that in $t = 1, 2, \ldots, n$ values of the process are fixed is considered. We partition the vector into two sub-vectors

$$\vec{\xi}_1 = (\xi_{1+1/2}, \ldots, \xi_{(n-1)+1/2})^T, \quad \vec{\xi}_2 = (\xi_1, \ldots, \xi_n)^T.\ l \tag{6.3}$$

The corresponding correlation matrices of vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ have the form:

$$R_{11} = \left\| \begin{array}{cccc} 1 & r_1 & \ldots r_{n-2} & \\ r_1 & 1 & \ldots r_{n-3} & \\ \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ r_{n-2} & r_{n-3} & \ldots & 1 \end{array} \right\|, \quad R_{11} = \left\| \begin{array}{cccc} 1 & r_1 & \ldots r_{n-1} & \\ r_1 & 1 & \ldots r_{n-2} & \\ \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ r_{n-1} & r_{n-2} & \ldots & 1 \end{array} \right\|, \tag{6.4}$$

Mutual correlation matrix $R_{12} = M\vec{\xi}_1\vec{\xi}_2^T$ of vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ of dimension $n \times (n-1)$ is given by

$$R_{12} = \left\| \begin{array}{cccc} r_{1/2} & r_{1/2} & r_{1+1/2} & \ldots r_{(n-2)+1/2} \\ r_{1+1/2} & r_{1/2} & r_{1/2} & \ldots r_{(n-3)+1/2} \\ \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ r_{(n-2)+1/2}\, r_{(n-3)+1/2} & & \ldots & r_{1/2} \end{array} \right\|, \tag{6.5}$$

As noted in Section 1.2, in accordance with the algorithm discussed in [15], if the normal vectors and in aggregate form vector $\vec{\xi} = (\vec{\xi}_1^T, \vec{\xi}_2^T)^T$ with an average

$$M\vec{\xi} = \vec{\mu} = (\vec{\mu}_1^T, \vec{\mu}_2^T)^T$$

and the joint covariance matrix

$$R = M(\vec{\xi} - \overleftarrow{\mu})(\vec{\xi} - \overleftarrow{\mu})^T = \left\| \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right\|,$$

where

$$M\vec{\xi}_1 = \vec{\mu}_1, \quad M\vec{\xi}_2 = \vec{\mu}_2,$$
$$R_{11} = M(\vec{\xi}_1 - \vec{\mu}_1)(\vec{\xi}_1 - \vec{\mu}_1)^T, \quad R_{21}^T = R_{12} = M(\vec{\xi}_1 - \vec{\mu}_1)(\vec{\xi}_2 - \vec{\mu}_2)^T,$$
$$R_{22} = M(\vec{\xi}_2 - \vec{\mu}_2)(\vec{\xi}_2 - \vec{\mu}_2)^T,$$

then the vector $\vec{\xi}_1$ for a given $\vec{\xi}_2$ constructs in accordance with (**??**) in the form

$$\vec{\xi}_1 = \vec{\mu}_1 + R_{12}R_{22}^{-1}(\vec{\xi}_2 - \vec{\mu}_2) + A_1\vec{\varphi}_1, \tag{6.6}$$

where $\vec{\varphi}_1 = (\varphi_{q+1}, \ldots, \varphi_n)^T$ are independent of one another and of $\vec{\xi}_2$ normal random variables with zero mean and unit variance, and a lower triangular matrix A_1 such that

$$A_1 A_1^T = R_{11.2} = R_{11} - R_{12}R_{22}^{-1}R_{21} \tag{6.7}$$
.

At realization of the algorithm (6.6) it is necessary to calculate the residual covariance matrix $R_{11.2}$, and then decompose it on the product of two triangular. followed by its decomposition product of two triangular. This requires time-consuming calculations, if the dimension of the corresponding matrices great.

In this regard, consider the following modification of the algorithm (6.6). Since the vector has the joint correlation matrix R, and $A_1\varphi_1$ in (6.6) is not dependent on $R_{12}R_{22}^{-1}\vec{\xi}_2$, then the expansion (6.7) can be avoided if use the following known property of the normal distribution (see, e.g., [1]).

Let

$$\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)^T = (\vec{\zeta}_1^T, \vec{\zeta}_2^T)^T,$$
$$\vec{\zeta}_1 = (\zeta_1, \ldots, \zeta_q)^T, \quad \vec{\zeta}_2 = (\zeta_{q+1}, \ldots, \zeta_n)^T$$

Is the normally distributed vector with zero mean and the joint covariance matrix

$$R = M\vec{\zeta}\vec{\zeta}^T = \left\| \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right\|,$$

where $\vec{\zeta}_1$ and $\vec{\zeta}_2$ are vectors with zero mean and covariance matrices $R_{11} = M\vec{\zeta}_1\vec{\zeta}_1^T$, $R_{12} = M\vec{\zeta}_1\vec{\zeta}_2^T$, $R_{22} = M\vec{\zeta}_2\vec{\zeta}_2^T$. If is not singular, then the vector

$$\vec{\eta} = \vec{\zeta}_1 - R_{12}R_{22}^{-1}\vec{\zeta}_2 \tag{6.8}$$

is normally distributed with zero mean and covariance matrix of the form (6.7). Really [32]

$$M(\vec{\zeta}_1 - R_{12}R_{22}^{-1}\vec{\zeta}_2)(\vec{\zeta}_1 - R_{12}R_{22}^{-1}\vec{\zeta}_2)^T =$$
$$= M\vec{\zeta}_1\vec{\zeta}_1^T - R_{12}R_{22}^{-1}M\vec{\zeta}_1\vec{\zeta}_2^T - R_{12}R_{22}^{-1}M\vec{\zeta}_2\vec{\zeta}_1^T +$$
$$+R_{12}R_{22}^{-1}M\vec{\zeta}_2\vec{\zeta}_2^T R_{22}^{-1}R_{21} =$$
$$= R_{11} - R_{12}R_{22}^{-1}R_{21}.$$

To simulate conditional realizations the vector$\vec{\xi}$ of the form (6.1) for fixed $\vec{\xi}_2$ from (6.3) we will use the connections (6.6), (6.8).

According to (6.6) vector $\vec{\xi}_1$ for a given$\vec{\xi}_2$ is constructed by transforming

$$\vec{\xi}_1 = R_{12}R_{22}^{-1}\vec{\xi}_2 + A_1\vec{\varphi}_1, \tag{6.9}$$

Where $\vec{\varphi}_1 = (\varphi_1, \ldots, \varphi_q)^T$ is the vector composed of independent normal variables with zero mean and unit variance, $R_{11}, R_{12}, R_{22}$are sub-blocks of the matrix R (6.2) and have the form (6.4), (6.5). Since the vector $\vec{\xi}$ has the joint correlation matrix R, and $A_1\vec{\varphi}_1$in (6.9) does not depend on$R_{12}R_{22}^{-1}\vec{\xi}_2$, then $\vec{\xi}_1$ by (6.8) can be represented as [9]

$$\vec{\xi}_1 = R_{12}R_{22}^{-1}\vec{\xi}_2 + \vec{\eta}_1 - R_{12}R_{22}^{-1}\vec{\eta}_2, \tag{6.10}$$

where the vectors

$$\vec{\eta}_1 = (\eta_{1+1/2}, \eta_{2+1/2}, \ldots, \eta_{(n-1)+1/2})^T, \quad \vec{\eta}_2 = (\eta_1, \ldots, \eta_n)^T \tag{6.11}$$

with correlation matricesR_{11} and R_{22}such that the vector

$$\vec{\eta} = (\eta_1, \eta_{1+1/2}, \ldots, \eta_{(n-1)+1/2}, \eta_n)^T$$

has normal distribution with zero mean, correlation matrix (6.1) and does not depend on $\vec{\xi}$.

Thus the representation (6.10) is equivalent to the representation (6.6).

We write the expression (6.10) in the identical form

$$\vec{\xi}_1 = R_{12}R_{22}^{-1}(\vec{\xi}_2 - \vec{\eta}_2) + \vec{\eta}_1. \tag{6.12}$$

Then, the interpolation algorithm for vector $\vec{\xi}_2$ in the points $t = 1 + 1/2, \ldots, (n-1) + 1/2$ has the form:

(1a) Independently of $\vec{\xi}$ normal vector $\vec{\eta} = (\eta_1, \eta_{1+1/2}, \ldots, \eta_{(n-1)+1/2}, \eta_n)^T$ with zero mean and correlation matrix R of the form (6.2) is simulated.

(2a) The vectors $\vec{\xi}$ and $\vec{\eta}$ are broken down into subvectors of the form (6.3), (6.11) respectively, and vector $\vec{\xi}_1$ at a fixed vector $\vec{\xi}_2$ is constructed using the transformation (6.12).

Since the correlation matrix R is a Toeplitz one, the transformation (6.12) can be represented in a recurrent form: for this in paragraph (1a) of the algorithm (1a) - (2a) we need to use the recursive procedure (3.4) - (3.6) for the simulation of Gaussian vectors with the correlation matrices of the Toeplitz type.

We construct the vector $\vec{\eta}$ with correlation matrix (6.1) by the scheme (3.4), (3.6). In order for the transformation (6.12) to be realized it is necessary to divide this vector into two subvectors $\vec{\eta}_1$ and $\vec{\eta}_2$ which are specified above. It remains to multiply the vector $\vec{\xi}_2 - \vec{\eta}_2 = \vec{x}$ by the regression matrix $R_{12}R_{22}^{-1}$ and add the result to the vector $\vec{\eta}_1$. The specific character of the matrices R_{12} and R_{22} allows us to do it recursively.

We first consider the product $R_{22}^{-1}\vec{x}$. The matrix R_{22} is the ToepHtz correlation matrix of dimension n. Therefore in order to realize this product we also use the representation, as in (3.1).

From (3.7) it follows that

$$R_n^{-1} = B_n^T D_n^{-1} B_n,$$

where the matrices B_n and D_n have the form (3.8) and (3.9). Thus, this representation is equivalent to the representation (3.7) where

$$D_n^{-1}B_n = \left\| \begin{array}{llll} 1 & & & \\ -\vec{b}^T[1]J_{(1)}\Big/d_1^2 & 1/d_1^2 & & \\ -\vec{b}^T[2]\Big/J_{(2)}d_1^2 & & 1/d_1^2 & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ -\vec{b}^T[n-1]J_{(n-1)}\Big/d_{(n-1)}^2 & & & 1/d_1^2 \end{array} \right\|,$$

$$B_n^T = \left\| \begin{array}{llllll} 1 & -J_{(1)}\vec{b}[1] & -J_{(2)}\vec{b}[2] & \cdots & -J_{(n-1)}\vec{b}[n-1] \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{array} \right\|,$$

We designate:

$$\vec{x}_k = (x_1, \ldots, x_k)^T, \quad \vec{y}_k = (y_1, \ldots, y_k)^T, \quad \vec{z}_k = (z_1, \ldots, z_k)^T.$$

When calculating d_k^2 and $\vec{b}[k]$ by the scheme (3.4), (3.6) we calculate $\vec{z}_n = B_n^T D_n^{-1} B_n \vec{x}_n$for the matrix R_n successively by the scheme:

$$\begin{array}{ll} y_1 = x_1, & \vec{z}_1 = y_1 \\ y_2 = \frac{x_2 - \vec{b}^T[1] J_1 \vec{x}_1}{d_1^2}, & \vec{z}_2 = \left\| \begin{array}{l} \vec{z}_1 - \vec{b}[1] y_2 \\ y_2 \end{array} \right\| \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & . \\ y_n = \frac{x_n - \vec{b}^T[n-1] J_{n-1} \vec{x}_{n-1}}{d_{n-1}^2}, & \vec{z}_n = \left\| \begin{array}{l} \vec{z}_{n-1} - \vec{b}[n-1] y_n \\ y_n \end{array} \right\| \end{array} \qquad (6.13)$$

It remains to multiply the vector $\vec{z}_n$ by the matrix R_{12}.

$$R_{12} R_{22}^{-1} \vec{x}_n = R_{12} \vec{z}_n.$$

We put

$$\vec{r}_k^T[n] = (\vec{p}^T[k] J_k, \vec{p}^T[n-k]), \quad k = 1, \ldots, n-1$$

and write the matrix R_{12} in the form

$$R_{12} = \left\| \begin{array}{l} \vec{r}_1^T[n] \\ \vec{r}_2^T[n] \\ \ldots\ldots\ldots \\ \vec{r}_{n-1}^T[n] \end{array} \right\|,$$

where $\vec{p}^T[k]$ is a row vector of dimension k in the form

$$\vec{p}^T[k] = (r_{0+1/2}, r_{1+1/2}, \ldots, r_{(k-1)+1/2}), \qquad (6.14)$$

When calculating $R_{12}\vec{z}$ a standard number of operations is needed to multiply the vector by the matrix. However, the size of memory for storing the matrix R_{12} with allowance for (6.14) is specified by the maximum dimension of the vector $\vec{p}^T[k]$ for $k = n-1$. The final result is the vector (6.12).

The above procedure is a special case of the more general procedure for modelling the conditional realizations of the vector $\vec{\xi}$ in the form

$$\vec{\xi} = (\xi_1, \xi_{1+1/p}, \ldots, \xi_{1+(p-1)/p}, \xi_2, \ldots, \xi_{n-1}, \xi_{n-1+1/p}, \ldots, \xi_{n-1+(p-)1/p}, \xi_n)^T, \qquad (6.15)$$

for fixed values of $\xi_1, \xi_2, \ldots, \xi_n$. As before the vector $\vec{\xi}$ is divided into two subvectors:

$$\begin{aligned}\vec{\xi_1} &= (\xi_{1+1/p}, \ldots, \xi_{1+(p-1)/p}, \ldots, \xi_{n-1+1/p}, \ldots, \xi_{n-1+(p-)1/p}, \xi_n)^T, \\ \vec{\xi_2} &= (\xi_1, \xi_2, \ldots, \xi_n)^T,\end{aligned} \tag{6.16}$$

The correlation matrix $M\vec{\xi}\vec{\xi}^T$ of the vector (1.15) is also considered to be the Toeplitz one and specified by its first row:

$$\begin{aligned}&(r_0, r_{0+1/p}, \ldots, r_{0+(p-1)/p}, r_1, \ldots, r_{1+(p-1)/p}, r_2, \ldots, \\ &, \ldots, r_{n-2+1/p}, \ldots, r_{n-2+(p-1)/p}, r_{n-1})^T,\end{aligned} \tag{6.17}$$

This row coincides with the first row of the matrix (6.1) for p = 2.

The procedure for modelling the conditional realizations of the vector (6.15) with the use of (6.12) does not differ radically from the previous one since the vector $\vec{z}_n$is constructed by the same formulae (6.13) and with the same dimensions of the vectors $\vec{x}_n, \vec{y}_n$. However, the vectors $\vec{\xi}$ and $\vec{\eta}$ are modelled with the correlation matrix that is specified by the row (6.17) in this case, and they are divided into the subvectors in accordance with (6.16).

The main difference is in the last operation of multiplication of the vector $\vec{z}_n$ by the matrix R_{12}. The matrix R_{12} in this case is a block one and has the form

$$R_{12} = \left\| \begin{array}{l} R_1[n] \\ R_2[n] \\ \cdots\cdots\cdots \\ R_{n-1}[n] \end{array} \right\|, \tag{6.18}$$

where $R_k[n]$ are blocks of dimension $(p-1) \times n$, which may be represented as

$$\begin{aligned}&R_k[n] = \\ &= \left\| \begin{array}{l} r_{k-(p-1)/p}\ r_{k-1-(p-1)/p}\ \cdots r_{0+(p-1)/p}\ \cdots r_{n-k-1+(p-1)/p} \\ r_{k-(p-2)/p}\ r_{k-1-(p-2)/p} \cdots r_{0+(p-2)/p} \cdots r_{n-k-1+(p-2)/p} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ r_{k-1/p} \qquad r_{k-1-1/p} \qquad \cdots r_{0+1/p} \qquad \cdots r_{n-k-1+1/p} \end{array} \right\|,\end{aligned}$$

Then in scalar notation the components of the vector $\vec{\xi_1} = R_{12}\vec{z}_n$ are calculated by the formula

$$\begin{aligned}\xi_{k+i/p} = \sum_{j=1}^{k+1} r_{|k+1-j|-(p-i)/p} z_j + \sum_{j=k+2}^{n} r_{|k+1-j|-(p-i)/p} z_j, \\ k = 1, \ldots, n-1, \quad i = 1, \ldots, p-1,\end{aligned} \tag{6.19}$$

where r_{h+q}, $h = 0, \cdots, n-2$, $q = 1/p, \ldots, (p-1)/p$ are the entries of the first row (6.17) in the correlation matrix of the vector $\vec{\xi}$. In order to choose the elements r_{s-1} of the first sum in (6.19) from the row (6.17), where

$$s = 1, \cdots, n-1, \; l = 1/p, \ldots, (p-1)/p,$$

we can use the identity $r_{s-1} = r_{s-1+(1-l)}$.

Note that the recursive procedures for modelling the conditional realizations of the Gaussian homogeneous fields at the nodes of a regular grid, given the values at the nodes of a rarefied regular grid, are constructed in a similar way, using the vector algorithms for simulating the Gaussian processes [27,34].

The above algorithms are essentially algorithms modeling errors of optimal interpolation of Gaussian stationary processes with rare nodes of a regular grid nodes in a dense grid. Generalization to the case of non-Gaussian processes can be made by the 'inverse function' method [31].

2. STATIONARY GAUSSIAN VECTOR SEQUENCES

2.1. Simulation of stationary Gaussian vector sequences and discrete spatial fields with a given correlation structure

Let us consider an algorithm of simulating stationarily correlated Gaussian p-vectors $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ with a given real valued covariance matrix $R_{(n)}$ of the block-stationary (or block-Toeplitz) type

$$R_{(n)} = \left\| \begin{array}{cccc} R_0 & R_1 & \cdots & R_{n-1} \\ R_1^T & R_0 & \cdots & R_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{n-1}^T & R_{n-2}^T & \cdots & R_T \end{array} \right\| \tag{1.1}$$

where $R_k,\ k = 0, 1, \ldots, n-1$ are $p \times p$-matrices. The positive definite real-valued matrix $R_{(n)}$ is symmetric, but its block elements may be both symmetric and non-symmetric. In practice, for example, when joint stationary series of weather elements are simulated, matrices R_n are frequently non-symmetric (their symmetry is approximately admissible only in some special cases). The sequence of vectors $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ may be interpreted as a random Gaussian field defined on any regular plane lattice involving $n \times p$ points.

In the case of an arbitrary stationary Gaussian field the problem is reduced to simulating stationary Gaussian vector sequence $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ corresponding to covariance matrix $R_{(n)}$ of type (1.1) with non-symmetric blocks R_n. We note that in the general case in simulating non-stationary Gaussian fields on the regular plane lattice with an arbitrary covariance $np \times np$-matrix R, it is expedient to use algorithm (1.2.2), (1.2.4) provided that the dimension of the matrix R is not too great.

Below some approximate approaches to the simulation of such fields will be also considered.

If the sequence $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ is interpreted as a homogeneous field with correlations independent of a parallel shift of the corresponding mesh points, then the blocks R_k of the matrix $R_{(n)}$ in the general ease are non-symmetric, but have some specific properties, namely, they are of the Toeplitz structure.

And if the field $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ is homogeneous isotropic, i.e., correlations depend only on the distance between mesh points, but not on their parallel shift and direction, then matrices R_n are not only of the Toeplitz type and symmetric, but also have some other additional features. In this Section the method of "conditional expectations", considered above, for the vector Gaussian sequences is generalized. The simulation scheme and calculation algorithms for the matrix of regression coefficients do not differ greatly from the corresponding algorithms for the scalar case, but have some characteristic properties due to the non-commutativity of the matrix multiplication. Some variants of algorithms for the case of stationary, homogeneous, and also homogeneous isotropic Gaussian fields on a finite lattice will be also considered. The results are based on the use of the well-known multi-channel algorithm of Levinson [34] as well as on the results, stated in [8,27].

Let us consider a block covariance matrix $\tilde{R}_{(n)}$, connected with matrix (1) by the relation

$$\tilde{R}_{(n)} = J_{(n)} R_{(n)} J_{(n)},$$

where

$$J_{(n)} = \left\| \begin{array}{ccc} 0 & \cdots 0 & I \\ 0 & \cdots I & 0 \\ \cdots & \cdots & \cdots \\ I & \cdots & 0 \end{array} \right\| \tag{1.2}$$

is a block permutation matrix, and I_p is the identity $p \times p$-matrix.

In the general case the non-symmetric blocks $\tilde{R}_k,\ k = 1, \ldots, n-1$ of the matrix $\tilde{R}_{(n)}$ are connected with non-symmetric blocks R_k of the matrix $R_{(n)}$ by the relation $\tilde{R}_k = R_k^T$ (and also $R_{(n)}^T = R_{(n)},\ \tilde{R}_{(n)}^T = \tilde{R}_{(n)},\ R_{(n)}^T \neq \tilde{R}_{(n)}$). Thus, the matrix $\tilde{R}_{(n)}$ has the following form:

$$\tilde{R}_{(n)} = \left\| \begin{array}{cc} R_1 \cdots \tilde{R}_{n-1} & \\ \tilde{R}_1^T \quad \tilde{R}_0 \cdots \tilde{R}_{n-2} & \\ \cdots\cdots\cdots & \\ \tilde{R}_{n-1}^T \quad \tilde{R}_{n-2}^T \cdots \tilde{R}_0 & \end{array} \right\| = \left\| \begin{array}{cc} R_0 \quad R_1^T \cdots & R_{n-1}^T \\ R_1 \quad R_0 \cdots & R_{n-2}^T \\ \cdots\cdots\cdots & \\ R_{n-1} \quad R_{n-2} \cdots & R_0 \end{array} \right\|$$

If $R_k \neq R_k^T$, i.e., the matrices R_k are non-symmetric, the matrix $\tilde{R}_{(n)}$ is of primary importance for the construction of the simulation algorithm.

To simulate the vectors $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$, the method of "conditional expectations" is used. The vectors $\vec{\xi}_k,\ k = 1, \ldots, n$ are calculated recursively by

the scheme

$$\begin{aligned}
&\vec{\xi}_1 = C_0\vec{\varphi}_1\\
&\vec{\xi}_2 = \vec{B}^T[1]J_{(1)}\vec{\xi}_{(1)} + C_1\vec{\varphi}_2\\
&\cdots\cdots\cdots\cdots\\
&\vec{\xi}_n = \vec{B}^T[n-1]J_{(n-1)}\vec{\xi}_{(n-1)} + C_{n-1}\vec{\varphi}_n,
\end{aligned} \tag{1.3}$$

where $\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_n$ are independent Gaussian p-dimensional vectors,

$$M\vec{\varphi}_k\vec{\varphi}_k^T = I_p, \quad \vec{\xi}_{(k)} = (\vec{\xi}_1^T, \ldots, \vec{\xi}_k^T)^T, \quad k = 1, \ldots, n,$$

$$\vec{B}[k] = (B_1^T[k], \ldots, B_k^T[k])^T, \quad k = 1, \ldots, n-1,$$

$B_i[k]$ are $p \times p$-matrices, and C_i are lower triangular $p \times p$ matrices. Numerical implementation of scheme (1.3) requires calculation of the matrix $\vec{B}[k]$ and the residual covariance matrix $Q_k = C_kC_k^T$ at each step. One of the possible methods for calculation of the matrix $\vec{B}[k]$ is the solution of the equation

$$\tilde{R}_{(k)}\vec{B}[k] = \vec{R}_k.$$

Methods for solution of systems $A\vec{x} = \vec{b}$ with block Toeplitz matrices are well known nowadays. Various modifications of these methods are described in [195]. As to the matrix Q_k, it is connected with $R_{(k)}$ and $B[k]$ by the following relation

$$Q_k = R_0 - \vec{B}^T[k]\tilde{R}_{(k)}\vec{B}[k] \tag{1.4}$$

where $\tilde{R}_{(n)} = J_{(n)}R_{(n)}J_{(n)}$. Calculations of Q_k by formula (1.4) require much computer costs, therefore a special recurrent procedure for calculating the matrices $B[k]$ and Q_k is considered.

Let

$$\begin{aligned}
&\tilde{R}_{(k)}^{-1}\vec{R}_{(k)} = \vec{B}[k], \quad \tilde{R}_{(k+1)}^{-1}\vec{R}_{k+1} = \vec{B}[k+1],\\
&R_{(k)}^{-1}\tilde{\vec{R}}_{(k)} = \tilde{\vec{B}}[k],\\
&R_{(k+1)}^{-1}\tilde{\vec{R}}_{k+1} = \tilde{\vec{B}}[k+1],
\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}
&\vec{B}[k] = (B_1^T[k], \ldots, B_k^T[k])^T,\\
&\vec{B}[k+1] = (B_1^T[k+1], \ldots, B_{k+1}^T[k+1])^T,\\
&\tilde{\vec{B}}[k] = (\tilde{B}_1^T[k], \ldots, \tilde{B}_k^T[k])^T,\\
&\tilde{\vec{B}}[k+1] = (\tilde{B}_1^T[k+1], \ldots, \tilde{B}_{k+1}^T[k+1])^T,\\
&\tilde{R}_{(k)} = J_{(k)}R_{(k)}J_{(k)}, \quad \tilde{R}_{(k+1)} = J_{(k+1)}R_{(k+1)}J_{(k+1)},\\
&\vec{R}_k = (R_1^T, \ldots, R_k^T)^T, \quad \vec{R}_{k+1} = (R_1^T, \ldots, R_{k+1}^T)^T,\\
&\tilde{\vec{R}}_k = (R_1^T, \ldots, R_k^T)^T, \quad \tilde{\vec{R}}_{k+1} = (R_1^T, \ldots, R_{k+1}^T)^T,
\end{aligned} \tag{1.6}$$

$$R_{(k+1)} = \left\| \begin{array}{cc} R_{(k)} & J_{(k)}\vec{R}_k \\ \vec{R}_k^T J_{(k)} & R_0 \end{array} \right\|$$

From (1.5) it follows that

$$R_{(k)}^{-1} J_{(k)} \vec{R}_k = J_{(k)} \vec{B}[k], \quad \tilde{R}_{(k)}^{-1} J_{(k)} \tilde{\vec{R}}_k = J_{(k)} \tilde{\vec{B}}[k]. \tag{1.7}$$

Using designations (1.6) by analogy with a scalar case let us consider the inverse block matrices for $R_{(k+1)}$ and $\tilde{R}_{(k+1)}$:

$$R_{(k+1)}^{-1} = \left\| \begin{array}{l} R_{(k)}^{-1} + R_{(k)}^{-1} J_{(k)} \vec{R}_k Q_k^{-1} \vec{R}_k^T J_{(k)} R_{(k)}^{-1} \vdots - R_{(k)}^{-1} J_{(k)} \vec{R}_k Q_k^{-1} \\ -Q_k^{-1} \vec{R}_k^T J_{(k)} R_{(k)}^{-1} \vdots Q_k^{-1} \end{array} \right\| \tag{1.8}$$

$$\tilde{R}_{(k+1)}^{-1} = \left\| \begin{array}{l} \tilde{R}_{(k)}^{-1} + \tilde{R}_{(k)}^{-1} J_{(k)} \tilde{\vec{R}}_k \tilde{Q}_k^{-1} \tilde{\vec{R}}_k^T J_{(k)} \tilde{R}_{(k)}^{-1} \vdots - \tilde{R}_{(k)}^{-1} J_{(k)} \tilde{\vec{R}}_k \tilde{Q}_k^{-1} \\ -\tilde{Q}_k^{-1} \tilde{\vec{R}}_k^T J_{(k)} \tilde{R}_{(k)}^{-1} \vdots \tilde{Q}_k^{-1} \end{array} \right\|,$$

where

$$Q_k = R_0 - \vec{R}_k^T \tilde{R}_{(k)}^{-1} \vec{R}_k,$$

$$\tilde{Q}_k = R_0 - \tilde{\vec{R}}_k^T R_{(k)}^{-1} \tilde{\vec{R}}_k.$$

Then $\vec{B}[k+1]$ and Q_k are related to $\vec{B}[k]$ in the following way:

$$\begin{array}{l} B_1^T[1] = R_1^T R_0^{-1}, \quad \tilde{B}_1^T[1] = R_1 R_0^{-1}, \\ Q_0 = R_0, \quad Q_0 = \tilde{R}_0 = R_0, \\ (B_1^T[k+1], \ldots, B_k^T[k+1]) = \vec{B}^T[k] - B_{k+1}^T[k+1] \vec{\tilde{B}}^T[k] J_{(k)}, \\ (\tilde{B}_1^T[k+1], \ldots, \tilde{B}_k^T[k+1]) = \vec{\tilde{B}}^T[k] - \tilde{B}_{k+1}^T[k+1] \vec{B}^T[k] J_{(k)}, \\ B_{k+1}[k+1] = \tilde{Q}_k^{-1} (R_{K+1} - \vec{\tilde{R}}_k^T J_{(k)} \vec{B}[k]), \\ \tilde{B}_{k+1}[k+1] = Q_k^{-1} (R_{K+1} - \vec{R}_k^T J_{(k)} \vec{\tilde{B}}[k]), \\ Q_k = R_0 - \vec{R}_k^T \vec{B}[k], \quad \tilde{Q}_k = \tilde{R}_0 - \vec{\tilde{R}}_k^T \vec{\tilde{B}}[k], \\ C_k C_k^T = Q_k, \\ k = 1, \ldots, n-1. \end{array} \tag{1.9}$$

It is possible to improve algorithm (1.9) if Q_k, $\tilde{Q}_k$ and $\tilde{B}_{k+1}[k+1]$ are calculated by formulas

$$
\begin{aligned}
&Q_k = Q_{k-1} - B_k^T[k]\tilde{Q}_{k-1}B_k[k],\\
&\tilde{Q}_k = \tilde{Q}_{k-1} - \tilde{B}_k^T[k]Q_{k-1}\tilde{B}_k[k],\\
&\tilde{B}_{k+1}[k+1] = Q_k^{-1}B_{k+1}^T[k+1]\tilde{Q}_k.
\end{aligned}
\tag{1.10}
$$

Let us consider some properties of algorithm (1.3), (1.9). It follows from the positive definition of the matrix $R_{(n)}$ that $R_0, R_{(1)}, \ldots, R_{(n-1)}$ and also $R_0^{-1}, R_{(1)}^{-1}, \ldots, R_{(n-1)}^{-1}$ are positive definite submatrices. Then because of (1.8), $Q_1, \ldots, Q_{n-1}$ will be positive definite matrices. As to the matrix Q_n, it will be positive definite provided the extended matrix $R_{(n+1)}$ is positive definite too. In addition, the following relation is valid:

$$
\det(R_{(n)}) = \det(Q_1)\cdot\det(Q_2)\cdots\det(Q_{n-1}) \tag{1.11}
$$

and

$$
\det(R_0) = \det(Q_0) \geq \det(Q_1) \geq \cdots \geq \det(Q_{n-1}). \tag{1.12}
$$

It is not difficult to show this by induction with the help of the well-known relation

$$
\det(R_{(n)}) = \det(R_{n-1})\det(R_0 - \vec{R}_{n-1}^T R_{(n-1)}^{-1}\vec{R}_{n-1})
$$

and using the positive definiteness of the matrices $R_0, R_{(1)}, \ldots, R_{(n-1)}$. It is clear that relations (1.11), (1.12) are also true for the matrix $\tilde{R}_{(n)}$.

It follows from (10) and Lemma 1 that eigenvalues of the matrices $B_1[1], \ldots, B_{n-1}[n-1]$ and those of the matrices $\tilde{B}_1[1], \ldots, \tilde{B}_{n-1}[n-1]$ modulo are less than unit.

Let C_k be a lower triangular matrix with positive diagonal elements which are uniquely defined by the condition

$$
C_k C_k^T = Q_k, \quad k = 0, \ldots, n-1.
$$

Let

$$
R_{(n)}^{-1} = T_{(n)}^T T_{(n)}, \quad T_{(n)} = C_{(n)}^{-1}B_{(n)}, \tag{1.13}
$$

where

$$
C_{(n)} = \left\| \begin{array}{cccc}
C_0 & 0 & \cdots & 0\\
0 & C_1 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots\\
0 & 0 & \cdots & C_{n-1}
\end{array} \right\|,
$$

(1.14)

$$B_{(n)} = \left\| \begin{array}{cccc} I_p & 0 & \cdots & 0 \\ -\vec{B}^T[1]J_{(1)} & I_p & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\vec{B}^T[n-1]J_{(n-1)} & & & I_p \end{array} \right\|,$$

Inversion of the left and right parts of equation (1.13) gives

$$R_{(n)} = T_{(n)}^{-1} T_{(n)}^{T-1} = B_{(n)}^{-1} C_{(n)} C_{(n)} B_{(n)}^{T-1},$$

where $B_{(n)}^{-1} C_{(n)}$ is the lower triangular block matrix. Let us consider the transformation

$$\vec{\xi}_{(n)} = B_{(n)}^{-1} C_{(n)} \vec{\varphi}_{(n)}, \tag{1.15}$$

where $\vec{\xi}_{(n)} = (\vec{\xi}_1^T, \ldots, \vec{\xi}_n^T)^T$, and $\vec{\varphi}_{(n)} = (\vec{\varphi}_1^T, \ldots, \vec{\varphi}_n^T)^T$ is a Gaussian vector with the block correlation matrix $I_{(n)}$ of the form

$$I_{(n)} = \left\| \begin{array}{cccc} I_p & 0 & \cdots & 0 \\ 0 & I_p & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_p \end{array} \right\|. \tag{1.16}$$

Then

$$\begin{array}{l} M\vec{\xi}_{(n)}\vec{\xi}_{(n)}^T = B_{(n)}^{-1} C_{(n)} M\vec{\varphi}_{(n)}\vec{\varphi}_{(n)}^T C_{(n)}^T (B_{(n)}^T)^{-1} = \\ = B_{(n)}^{-1} C_{(n)} C_{(n)}^T (B_{(n)}^{-1})^T = (B_{(n)}^T (C_{(n)}^{-1})^T C_{(n)}^{-1} B_{(n)})^{-1} = \\ = (R_{(n)}^{-1})^{-1} = R_{(n)}. \end{array}$$

Rewrite (1.15) in the form

$$B_{(n)}\vec{\xi}_{(n)} = C_{(n)}\vec{\varphi}_{(n)}$$

In view of relations (1.14), this transformation has the same effect as scheme (1.3).

The matrices $\vec{B}[k]$ and $\tilde{Q}_k$ in algorithm (1.9) for scheme (1.3) play an auxiliary part for calculating matrices $\vec{B}[k]$ and Q_k. Nevertheless, matrices $\tilde{\vec{B}}$ and Q_k may be used, if it is necessary to simulate the vector $\vec{\xi}_0$, provided the sequence $\vec{\xi}_1, \vec{\xi}_1, \ldots, \vec{\xi}_k$ has been simulated. To see this, let $\vec{\xi}_{(k)} = (\vec{\xi}_1^T, \ldots, \vec{\xi}_k^T)^T$ be a Gaussian sequence with the block covariance matrix $R_{(k)}$. Let us simulate $\vec{\xi}_0$ such that the sequence $(\vec{\xi}_0^T, \vec{\xi}_1^T, \ldots, \vec{\xi}_k^T)^T$ have

the given covariance matrix $R_{(k+1)}$. To this end let us consider the following transformation.

$$\vec{\xi}_0 = (\tilde{B}_1^T[1], \ldots, \tilde{B}_k^T[k])\vec{\xi}_{(k)} + \tilde{C}_k\vec{\varphi}_0, \tag{1.17}$$

where $\tilde{C}_k$ is the lower triangular $p \times p$-matrix such that $\tilde{C}_k\tilde{C}_k^T = \tilde{Q}_k$ and $\vec{\varphi}_0$ is a Gaussian vector independent of the vector $\vec{\xi}_{(k)}$ with non-correlated components. Multiplying (1.17) by $\vec{\xi}_k^T$ from the right and taking into account the fact that $M(\vec{\xi}_0, \vec{\xi}_k^T) = (R_1, \ldots, R_k)$, we obtain

$$R_{(k)}\tilde{\vec{B}}[k] = \tilde{\vec{R}}_k,$$

which corresponds to the third relation in (1.5). Then, multiplying (1.17) by $\vec{\xi}_0^T$ from the right and requiring $M\vec{\xi}_0\vec{\xi}_0^T = R_0$, we have (see, also [98])

$$\tilde{Q}_k = R_0 - \tilde{\vec{R}}_k^T\tilde{\vec{B}}[k]. \tag{1.18}$$

Further this property and also those considered earlier will be used for the simulation of vector autoregressive processes with a given correlation structure and, also, for simulation of discrete Gaussian random fields on the regular lattice.

Numerical experiments carried out for various matrices $R_{(n)}$ show that algorithm (9) in many cases is numerically stable. Nevertheless, in some cases this algorithm is unstable. Instability is revealed in that at some step of calculations the matrices Q_k and $\tilde{Q}_k$ become negative definite and, thus, the results of the calculations loose the probabilistic sense since the matrix Q_k is a residual covariance matrix in forecasting $\vec{\xi}_{k+1}$ by $\vec{\xi}_1$ provided that the vectors $\vec{\xi}_k, \ldots, \vec{\xi}_2$ are fixed. In these cases instead of $\vec{\xi}_1, \ldots, \vec{\xi}_n$ it seems better to restrict ourselves to simulation of the vector sequence $\vec{\zeta}_1, \ldots, \vec{\zeta}_n$ with the joint covariance matrix $G_{(n)}$ connected with the matrix $R_{(n)}$ by the relation

$$G_{(n)} = \varepsilon I_{(n)} + (1 - \varepsilon)R_n, \tag{1.19}$$

where ε is a number from the interval $(0, 1)$, and $I_{(n)} = I_{np}$ is the identity $np \times np$-matrix. The parameter ε may be always selected so that algorithm (1.9) for the matrix $G_{(n)}$ be numerically stable. The fact that the minimal eigenvalue of the matrix $G_{(n)}$ is more than ε for an arbitrary n is responsible for this. Experiments show that in many cases, rather small values of ε (of order 10^{-4}) are sufficient for stability of algorithm (1.9). From the point of

view of many applications the errors of such an order are fairly acceptable: it means that the correlation $y_{|i-j|,|k-e|}$ between the components ζ_{ik} and ζ_{je} of the simulated vectors $\vec{\zeta}_i$ and $\vec{\zeta}_j$ are distinguished from the correlation $r_{|i-j|,|k-e|}$ not more than by ε, both in the absolute and in the relative value.

Note, that the vector $\vec{\zeta}_{(n)}$ with covariance matrix (1.19) is connected with the vector $\vec{\xi}_{(n)}$ by the relation

$$\vec{\zeta}_k = \sqrt{1-\varepsilon}\,\vec{\xi}_k + \sqrt{\varepsilon}\,\vec{\eta}_k, \quad k = 1, \ldots, n,$$

where $\vec{\eta}_1, \vec{\eta}_2, \ldots, \vec{\eta}_n$ are mutually independent of $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_n$ Gaussian random vectors with zero means and the identity correlation matrices. Thus, the vector $\vec{\zeta}_{(n)}$ is derived from the vector $\vec{\xi}_{(n)}$ by imposing the Gaussian noise with a small variance.

For the realization of algorithm (1.9) it is necessary to store in the computer memory the block rows of matrices $R_{(n)}$ and $\tilde{R}_{(n)}$, matrices $\tilde{B}[k]$ and $\tilde{\mathcal{B}}[k]$ whose maximal number of matrix blocks is equal to $n-1$, and also, one $p \times p \times (n-1)$-matrix to store results of intermediate calculations.

This circumstance restricts the possibility of the algorithm. If it is required to simulate an ensemble of vector process realizations, it can be constructed by accumulation of vector $\vec{\xi}_k$ realizations at each step by k. At the decrease of p, naturally, the length of a simulated sequence may be increased. In case $p = 1$, $\vec{\xi}_{(n)}$ denotes a sequence of n scalar values, and in this case, the algorithm is most efficient.

2.1.1.Simulation of homogeneous discrete Gaussian fields.

The vector sequence $\vec{\xi}_{(n)} = (\vec{\xi}_1^T, \ldots, \vec{\xi}_n^T)^T$ can be interpreted as a discrete Gaussian field at the regular finite difference grid in R^2. Let i and j be integer coordinates of grid points, $i = 1, \ldots, n$, $j = 1, \ldots, p$, then j-th component of the vector $\vec{\xi}_i$ from the sequence $\vec{\xi}_{(n)}$ is a value of a random field at a grid point with the coordinates i and j. If the field is homogeneous, then the covariances between values of this field at two grid points do not depend on a parallel shift of grid points and satisfy the relation

$$M\xi_{i,j}\ \xi_{i_1,j_1} = M\xi_{i+k,j+l}\ \xi_{i_1+k,j_1+l} = r_l^k. \tag{1.20}$$

The covariance matrix $R_{(n)}$ of the field $\vec{\xi}_{(n)}$ in this case has the block structure of type (1), and its blocks R_k are non-symmetric $p \times p$-matrices of the Toeplitz type

$$R_k = \left\| \begin{array}{cccc} r_0^k & r_0^k & \cdots & r_{p-1}^k \\ r_{-1}^k & r_0^k & \cdots & r_{p-2}^k \\ \cdots & \cdots & \cdots & \cdots \\ r_{-(p-1)}^k & r_{-(p-2)}^k & \cdots & r_0^k \end{array} \right\|, \tag{1.21}$$

In literature, the block Toeplitz matrix $R_{(n)}$ with blocks of type (21) are termed twice-Toeplitz, or Toeplitz-Toeplitz matrix [98].

Let us consider a block permutation matrix in the form

$$\gamma_{(n)} = \left\| \begin{array}{cccc} J_p & 0 & \cdots & 0 \\ 0 & J_p & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_p \end{array} \right\|,$$

and present its main properties:

$$\begin{array}{ll} \gamma_{(n)}\gamma_{(n)} = I_{(n)}, & \gamma_{(n)}^{-1} = \gamma_{(n)}, \\ \gamma_{(n)}J_{(n)} = J_{np}, & J_{(n)}\gamma_{(n)} = J_{np}, \end{array}$$

where J_{np} is the np-dimensional permutation matrix of type (1.2.11). For the twice-Toeplitz covariance matrix $R_{(n)}$ the following relation takes place

$$J_{(n)}R_{(n)}J_{(n)} = \gamma_{(n)}R_{(n)}\gamma_{(n)} = \tilde{R}_{(n)}\ . \tag{1.23}$$

The latter relation is easy to check by multiplying the corresponding matrices. Let us now consider the equations from (1.5)

$$\tilde{R}_{(k)}\vec{B}[k] = \vec{R}_k, \quad R_{(k)}\tilde{\vec{B}}[k] = \tilde{\vec{R}}_k, \tag{1.24}$$

and rewrite the second equation in the identical form

$$\gamma_{(k)}R_{(k)}\gamma_{(k)}\gamma_{(k)}\tilde{\vec{B}}[k]J_p = \gamma_{(k)}\tilde{\vec{R}}_k J_p. \tag{1.25}$$

It is obvious that for a homogeneous field the relation $\gamma_{(k)}\tilde{\vec{R}}_k J_p = \vec{R}_k$ takes place. Then it follows from relations (1.23), (1.24) and non-singularity of the matrix $R_{(n)}$ that

$$\tilde{\vec{B}}[k] = \gamma_{(n)}\vec{B}[k]J_p.$$

As a result, the algorithm for calculation of the matrix regression coefficients $B_i^T[k]$ and the residual covariance matrix Q_i may be written down in the following form

$$\begin{array}{l}
B_1^T[1] = R_1^T R_0, \\
Q_0 = R_0, \quad C_0 C_0^T = Q_0, \\
(B_1^T[k+1], \ldots, B_k^T[k+1]) = \vec{B}^T[k] - B_{k+1}^T[k+1]J_p\vec{B}^T[k]J_{kp}, \\
B_{k+1}[k+1] = J_p Q_k^{-1} J_p (R_{K+1} - J_p\vec{R}_k^T J_{kp}\vec{B}[k]), \\
Q_k = Q_{k-1} - B_k^T[k]J_p Q_{k-1} J_p \vec{B}[k], \\
C_k C_k^T = Q_k^T, \\
k = 1, \ldots, n-1.
\end{array} \tag{1.26}$$

Note, that only symmetricity of the Toeplitz matrix along the lateral diagonal was used for obtaining these relations. Obviously, further simplifications of the algorithm are possible, if one takes into account other properties of the Toeplitz matrix, for example, its symmetricity along the lateral diagonal of the matrixes R_k. In this case the formulas (1.26) take the form

$$\begin{array}{l}
\vec{B}^T[1] = B_1^T[1] = R_1^T R_0^{-1}, \\
Q_0 = R_0, \quad C_0 C_0^T = Q_0, \\
(B_1^T[k+1], \ldots, B_k^T[k+1]) = \vec{B}^T[k] - B_{k+1}^T[k+1]\vec{B}^T[k]J_{(k)}, \\
B_{k+1}[k+1] = Q_k^{-1}(R_{K+1} - \vec{R}_k^T J_{(k)}\vec{B}[k]), \\
Q_k = Q_{k-1} - B_k^T[k]Q_{k-1}\vec{B}[k], \\
C_k C_k^T = Q_k^T, \\
k = 1, \ldots, n-1.
\end{array}$$

2.1.2.Simulation of homogeneous isotropic discrete Gaussian fields at the bounded grid.

Unlike the homogeneous field, the block R_k of the covariance matrix $R_{(n)}$ of the homogeneous isotropic field is not only Toeplitz's, but also symmetric. This is connected with the fact that along with (1.20), for the case of a homogeneous isotropic field the following relations, in particular, take place

$$r_l^k \;=\; r_{|l|}^{|k|}, \quad r_l^k \;=\; r_k^l. \tag{1.27}$$

It follows from (23) and (27), also, that for any submatrix $R_{(1)} = R_0$, $R_{(2)}, \ldots, R_{(k)}, \ldots, R_{(n)}$ of a homogeneous and isotropic field the following equality is valid:

$$\gamma_{(k)} R_{(k)} \gamma_{(k)} = R_{(k)}. \tag{1.28}$$

It, also, follows from independence of covariances from shift and direction, that

$$J_{(k)} R_{(k)} J_{(k)} = R_{(k)}.$$

In this case relations follow from equality (1.23) by substitution of $\tilde{R}_{(k)}$ instead of $R_{(k)}$. Using this property and, also, properties of the matrices $\gamma_{(k)}$ and $J_{(k)}$, we may substantially simplify the algorithm of calculation of the regression matrices $B_1[k], \ldots, B_k[k]$ and, correspondingly, the residual matrices Q_k. Actually, the equation $\tilde{R}_{(k)} \vec{B}[k] = \vec{R}_k$ provided (1.27) takes the form $R_{(k)} \vec{B}[k] = \vec{R}_k$. Representing this equation in the equivalent form $\gamma_{(k)} R_{(k)} \gamma_{(k)} \gamma_{(k)} \vec{B}[k] J_p = \gamma_{(k)} \vec{R}_k J_p$, and taking into account, also, relation (28) the obvious equality $\gamma_{(k)} \vec{R}_k J_p = \vec{R}_k$, and non-singularity of the matrix $R_{(k)}$, we have

$$\gamma_{(k)} \vec{B}[k] J_p = \vec{B}[k]. \tag{1.29}$$

Thus, with allowance for relation (1.27), algorithm (1.9) for a homogeneous and isotropic field takes the form [127]

$$\begin{array}{l} B_1^T[1] = R_1 R_0^{-1}, \\ Q_0 = R_0, \\ (B_1^T[k+1], \ldots, B_k^T[k+1]) = \vec{B}^T[k] - B_{k+1}[k+1] \vec{B}^T[k] J_{(n)}, \\ B_{k+1}[k+1] = Q_k^{-1} (R_{k+1} - \vec{R}_k^T J_{(k)} \vec{B}[k]), \\ Q_k = R_0 - \vec{R}_k^T \vec{B}[k], \\ C_k C_k^T = Q_k, \\ k = 1, \ldots, n-1. \end{array} \tag{1.30}$$

The matrix Q_k with relation (1.10) taken into consideration is easier to calculate by formula

$$Q_k = Q_k - B_k^T[k] Q_k B_k[k].$$

At each step k it is sufficient to calculate only half of elements of each of the matrices $B_1[k], \ldots, B_k[k]$ by formula (1.30). Other elements of these matrices are determined by using relation (1.29).

2.2. Simulation of vector autoregressive sequences

We apply this algorithm to simulate stationary vector autoregressive processes with a given covariance structure. Consider the difference equations

$$\vec{\xi}_t = B_i^T[m]\vec{\xi}_{t-1} + \ldots + B_m^T[m]\vec{\xi}_{t-m} + \vec{u}_t, \tag{2.1}$$

where $\ldots, \vec{u}_{-1}, \vec{u}_0, \vec{u}_1, \ldots$ is a sequence of mutually independent p-vectors, which are also independent of $\vec{\xi}_t$ such that

$$M\vec{u}_t\vec{u}_t^T = Q_m,$$

and $B_1^T[m], \ldots, B_m^T[m]$ are $p \times p$ block matrices.

Rewrite (1) in the form:

$$\tilde{\vec{\xi}}_{(t)} = \Phi_{(m)}\tilde{\vec{u}}_{(t-1)} + \tilde{\vec{u}}_{(t)}, \tag{2.2}$$

where

$$\tilde{\vec{\xi}}_{(t)} = (\vec{\xi}_t^T, \ldots, \vec{\xi}_{t-p}^T)^T, \quad \tilde{\vec{u}}_{(t)} = (\vec{u}_t^T, \vec{0}^T, \ldots, \vec{0}^T)^T,$$

and $\Phi_{(m)}$ is a block Frobenius matrix of the form

$$\Phi_{(m)} = \left\| \begin{array}{ccccc} B_1^T[m] & B_2^T[m] & \cdots & B_{m-1}^T[m] & B_m^T[m] \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & o & \cdots & I & 0 \end{array} \right\| \tag{2.3}$$

Let the matrices $B_1^T[m], \ldots, B_m^T[m]$ in (2.3) be given. If the eigenvalues λ_k, $k = 1, \ldots, mp$ of the corresponding matrix $\Phi_{(m)}$ lie inside the unit circle,

then equation (2.1) has a unique stationary solution in the form of a mean square convergent series

$$\tilde{\vec{\xi}}_t = \sum_{\tau=0}^{\infty} \Phi_{(m)}^{\tau} \tilde{\vec{u}}_{t-\tau}. \tag{2.4}$$

Note that the block correlation function of process (1) is uniquely defined by the matrices $B_1^T[m], \ldots, B_m^T[m]$ and Q_m on the basis of a block variant of equation (1.5.3) with appropriate boundary conditions [92].

Here, as in the scalar case, it is assumed that not the matrices $B_1^T[m], \ldots, B_m^T[m]$, but the correlation function $R_0, R_1, \ldots, R_m$ (in this case a block matrix) is given. It is necessary to find the corresponding matrix coefficients $B_1^T[m], \ldots, B_m^T[m]$ and the residual covariance matrix Q_m in equation (2.1). The autoregressive process exists if the eigenvalues of the corresponding matrix (2.3) are less than unit.

For the case, if the submatrices R_k of matrix $R_{(m)}$ are symmetry we will show that if $B_1^T[m], \ldots, B_m^T[m]$ are solutions of the linear system

$$\tilde{R}_{(m)} \vec{B}[m] = \vec{R}_m$$

and the matrix $\tilde{R}_{m+1}$ (or, respectively, the matrix $R_{(m+1)}$) is positive definite, then equation (2.1) has a unique stationary solution representable in the form of (2.4). The initial values of the block correlation function will then coincide with the first block row of the matrix $R_{(m+1)}$. In the proof of this statement we use relations (2.1.9).

Lemma 4.. If $R_{(m)}$ is a positive definite matrix with symmetric blocks and R_k, and $\vec{B}[m]$ is the solution of the equation

$$R_{(m)} \vec{B}[m] = \vec{R}_m$$

then $\left|\lambda(\Phi_{(m)})\right| < 1$.

Proof of this statement is analogy to proof of Lemma 1.

Lemma 5. If $R_{(m+1)}$ is a positive definite matrix with symmetric blocks and R_k, and $\vec{B}[m]$ is the solution of the equation-

$$R_{(m)} \vec{B}[m] = \vec{R}_m$$

then $+1$ may not be an eigenvalue of the matrix $\Phi_{(m)}$.

Proof From the condition of positive definiteness of the matrix $R_{(m)}$ is follows that $R_0^{-1}, R_{(1)}^{-1}, \ldots, R_{(m-1)}^{-1}$ are positive definite matrices. Then the

matrices$Q_1, \dots Q_{m-1}$ are positive definite. The matrix Q_m is also positive definite. It is implied by the fact that $R_{(m+1)}$ is a positive definite matrix.

From (2.10) and symmetrisity of blocks of matrix $R_{(m+1)}$ follows that

$$Q_k = Q_{k-1} - B_k[k] Q_{k-1} B_k^T[k],$$

and from Lemma it follows that eigenvalues $\lambda(B_k[k]),\ k = 1, \dots, m$ of matrix $B_k[k]$ satisfy the condition

$$|\lambda(B_k[k])| < 1.$$

onside the summs

$$S_k^T = B_1^T[k] + \dots + B_k^T[k], \quad S_{k-1}^T = B_1^T[k-1] + \dots + B_{k-1}^T[k-1].$$

Using (2.9), it is not difficult to show that

$$S_k^T = S_{k-1}^T (I - B_k^T[k]).$$

Then

$$\det(S_k^T) = \det(S_{k-1}^T) \det(I - B_k^T[k]).$$

Using the known relation

$$\det\left(\Phi_m - \lambda I_{(m)}\right) = \det\left(\lambda^m - B_1^T[m]\lambda^{m-1} - \dots - B_m^T[m]\right),$$

and condition $|\lambda(B_k^t[k])| < 1$ by induction we can show that the value $+1$ is not eigenvalue of matrix $\Phi_{(m)}$.

Lemma 6. If $R_{(m+1)}$ is a positive definite matrix with symmetric blocks and $\vec{B}[m]$ is the solution of the equation

$$\tilde{R}_{(m)} \vec{B}[m] = \vec{R}_m,$$

then the value -1 is not an eigenvalue of the matrix.

The proof is analogous to the proof of Lemma 3.

Lemmas 4., 5. and 6. imply

Theorem 2. If $R_{(m+1)}$ is a positive definite matrix with symmetric blocksand $\vec{B}$ is the solution of the equation

$$\tilde{R}_{(m)} \vec{B}[m] = \vec{R}_m,$$

then $|\lambda(\Phi_{(m)})| < 1$.

Thus, the process, generated by the difference equation (1) with $B_1^T[m], \ldots, B_m^T[m]$ obtained by solving the system $\tilde{R}_{(m)}\vec{B}[m] = R_m$ under the condition $R_{(m+1)}$ being positive definite, is a stationary vector autoregressive process. The block covariance function R_h of this process for $h = 0, 1, \ldots, m$ coincides with the first block row of the matrix $R_{(m+1)}$. The blocks of the covariance function R_n at $h = m+1, m+2, \ldots$ are defined from the equation

$$R_h = \sum_{k=1}^{m} B_k^T[k] R_{h-k}.$$

The initial vectors $\vec{\xi}_1, \ldots, \vec{\xi}_m$ of the process $\vec{\xi}_t$ are calculated according to scheme (2.1.3), using (2.1.9) and (2.1.10).

The above-mentioned algorithms for block-Toeplitz matrices $R_{(m)}$ are intended for the simulation of random processes and fields having specific features such as stationary vector sequences, homogeneous and isotropic fields. Here we deal only with modelling the Gaussian processes and fields, although the above-considered transformations can be also used for simulation of non-Gaussian processes and fields. If covariance matrices do not possess any specific features, the corresponding field can be simulated in the manner discussed in the begining of this Chapter. Since for large-dimensional matrices these procedures require much computer costs it is possible to use other techniques of simplifying covariance matrices. If, for example, a correlation scale is much less than the spatial scale of a field, then one can make use of "tape" positive definite matrices. In this case algorithms can be essentially simplified, and simulation can be done on the grids with a large number of mesh points. Moving average models are a special case of these models. The latter can be used for the modelling of fields on finite and unlimited grids. In the case of an unlimited grid it is possible to use a spatial autoregressive model, where the values of a field at each mesh point are simulated by a certain number of the values of the field in the vicinity of this point. A detailed review of the issues connected with autoregressive models is presented in [18,27].

2.3. Stochastic model of periodically correlated time series

Consider a periodically correlated random process [11,13] of a discrete argument $\xi(t_i) = \xi_i$ (or a sequence) on a bounded interval at equidistant time

moments $t_1, \ldots, t_N$, $\Delta t = t_{i+1} - t_i$, with a period $T = p\Delta t$,where p is such that $T \leq t_N$. Its mathematical expectation $E\xi(t_i)$,variance $D\xi(t_i)$, and correlation function $R_\xi(t_i, t_j)$ satisfies relations (3.1):

$$\begin{gathered} E\xi(t_i + T) = E\xi(t_i), \\ D\xi(t_i + T) = D\xi(t_i), \\ R_\xi(t_i + T,\ t_j + T) = R_\xi(t_i,\ t_j). \end{gathered} \tag{3.1}$$

For simplicity we take $t_1 = 0$, and $\Delta t = 1$. In this case $T = p$.

The idea of simulation of Gaussian sequences satisfying these conditions belongs to V. A. Rozhkov and was first realized (as far as we know) in [3] in the form of a first-order autoregression vector model. The method was based on simulation of stationary linked Gaussian p-dimensional vectors $\vec{\xi}_i = (\xi_{1i},\ \xi_{2i}, \ldots,\ \xi_{pi})^T$, the set of these vectors can be represented for $i = 1, \ldots, n$ as a vector

$$\vec{\xi}_{(n)} = (\vec{\xi}_1^T,\ \vec{\xi}_2^T, \ldots,\ \vec{\xi}_n^T)^T \tag{3.2}$$

with zero mean and a given real covariation matrix of the block Toeplitz form

$$R_{(n)} = \left\| \begin{array}{cccc} R_0 & R_1 & ... & R_{n-1} \\ R_1^T & R_0 & ... & R_{n-2} \\ ... & ... & ... & ... \\ R_{n-1}^T & R_{n-2}^T & ... & R_0 \end{array} \right\|, \tag{3.3}$$

where $R_k,\ \ k = 0, \ldots,\ n-1$ are $p \times p$ matrices. The real positive definite matrix is symmetric, but its block elements may be asymmetric. Without loss of generality we can consider the normalized block covariation matrix $R_{(n)}$ The covariation matrix is obtained after multiplication of each component of vector (3.2) by the corresponding standard deviation.

Represent vector (3.2) in the form

$$\vec{\xi}_{(n)} = (\xi_{11},\ \xi_{21}, \ldots,\ \xi_{p1}, \xi_{12},\ \xi_{22}, \ldots,\ \xi_{p2}, \ldots,\ \xi_{1n},\ \xi_{2n}, \ldots, \xi_{pn})^T. \tag{3.4}$$

Show that the sequence of components of this vector is periodically correlated. The first two conditions from (3.1) are ensured by multiplication of each element of sequence (3.4) by the elements of the nonrandom periodic sequence of the corresponding standard deviations with the period $T = p$ and by addition of the nonrandom periodic sequence of the corresponding means with the same period to sequence (3.4). In order to verify the last

condition from (3.1), renumber the components of the vector (3.4) successively from 1 to np. Rewrite the vector in this enumeration in the following form:

$$\vec{\xi}_{(n)} = (\xi_1,\ \xi_2, \ldots,\ \xi_p, \xi_{p+1},\ \xi_{p+2}, \ldots,\ \xi_{2p}, \ldots,\ \xi_{(n-1)p+1},\ \xi_{(n-1)p+2}, \ldots, \xi_{np})^T.$$

Each component of this vector is representable in the form ξ_{kp+l}, $k = 0, \ldots,\ n-1,\quad l = 1, \ldots,\ p$, where $k+1$ is the number of a subvector in vector (3.4), l is the number of the component in this subvector. Then

$$E\xi_{kp+l+p}\xi_{k'p+l'+p} = E\xi_{(k+1)p+l}\xi_{(k'+1)p+l'} = E\xi_{kp+l}\xi_{k'p+l'}.$$

In this expression the latter equality is true because of the block Toeplitz structure of matrix (3.3). This fact obviously implies that the last equality from (3.1) holds.

Note that the specific character of the periodic correlation of sequence (3.4) is caused by the structure of blocks R_k of the matrix $R_{(n)}$.

The simulation algorithm of the sequence of vectors $\vec{\xi}_i$, $i = 1, \ldots, n$ with covariance matrix (3.3) is considered in this chapter (see (2.1.3),(2.1.9)).

In order to model infinite periodically correlated sequences

$$\xi_1,\ \xi_2, \ldots,\ \xi_p, \xi_{p+1},\ \xi_{p+2}, \ldots,\ \xi_{2p}, \ldots,\ \xi_{(n-1)p+1},\ \xi_{(n-1)p+2}, \ldots, \xi_{np}, \ldots$$

one can use a many-dimensional autoregression model of the order $n-1$ in the form

$$\vec{\xi}_t = B_1^T[n-1]\vec{\xi}_{t-1} + \ldots + B_{n-1}^T[n-1]\vec{\xi}_{t-(n-1)} + C_{n-1}\vec{\varphi}_t, \quad n = 1, 2, \ldots,$$

where $\vec{\xi}_1, \ldots,\ \vec{\xi}_{n-1}$ are used as initial vectors. If the stationarity properties hold for autoregression process [9], then relations (3.1) hold for it too.

3. NONGAUSSIAN MODELS

3.1. Piecewise constant random vectors and processes

textbf1 In this Section, we discuss the method for simulation of random vectors and processes on the basis of randomization of correlation matrices, suggested and investigated by G.A. Mikhailov. In literature this method is termed the "repetition" method. First, this approach was proposed in paper [23] "G.A. Mikhailov. On the "repetition" method for modelling of random vectors and processes (randomization of correlation matrices). Teoriya Veroyatnostey i ee Primeneniya (1974) 19, No. 4, pp. 873-878 [in Russian]" on an example of simulation of a two-dimensional vector $\vec{\xi} = (\xi_1, \xi_2)^T$ with the given one-dimensional distribution and correlation coefficient $r > 0$. In paper [23], this method was generalized to the case of n-dimensional vectors $\vec{\xi} = (\xi_1, \ldots, \xi_n)^T$ with the given correlation coefficients

$$r_{ik} > 0, \quad i, k = 1, \ldots, n$$

and the same one-dimensional distributions for all the components and, also, for the case of random processes $\xi(t)$ with the correlation functions of a special form. The main idea of the method may be illustrated on several simple examples [23].

1. Let $r_{ik} = r$ for $i \neq k$. Then for a correlation matrix $R = (r_{ik})$ the following representation is valid

$$R = rR_1 + (1 - r)R_0, \ , \tag{1.1}$$

where R_1 is a matrix, consisting of unities and R_0 is the identity matrix. From (1.1) follows a simple algorithm of simulation of a random vector: the same value for all components of the vector $\vec{\xi}$ is sampled with the probability r, and with the probability $1 - r$ these components are independently simulated with respect to the given one-dimensional distribution.

2. Let $r_{ik} = r^{|i-k|}$, $0 < r < 1$. It is easy to show that the vector $\vec{\xi}$ may be simulated by the following way: after simulation of the values $\xi_1, \ldots, \xi_i$, one puts $\xi_{i+1} = \xi_i$ with the probability r, otherwise, a value ξ_{i+1} is independently sampled $(i = 1, 2, \ldots, n - 1)$.

In paper [101] a more general case when elements of matrix R have the following form:

$$r_{ik} = p_0 r + \sum_{m=1}^{\infty} p_m r_m^{|i-k|}, \quad i \neq k, \tag{1.2}$$

was also considered, moreover, $p_m \geq 0$, $\sum_{m=0}^{\infty} p_m \leq 1$, $m = 0, 1, 2, \ldots$. From (1.2) follows the algorithm: with the probability p_0 a vector $\vec{\xi}$ is simulated in the same manner as in $\vec{1}$.; with the probability p_m it is simulated as in $\vec{2}$. for $r = r_m$; with the probability $p = 1 - \sum p_m$ values of the vector components are chosen independently.

Thus, a random vector may be easily simulated by the method of "repetition", the correlation matrix being represented (accurately or approximately) in the form of (1.2).

The algorithms $\vec{1}$. and $\vec{2}$. are easily generalized to the case of the corresponding random processes.

3. Let a correlation function $r(t, t')$ of a random process

$$r(t, t') = \begin{cases} 1 & \text{for} \quad t = t' \\ r & \text{for} \quad t \neq t', \end{cases}, \tag{1.3}$$

and a one-dimensional distribution, the same for all values t, be given. Then with the probability r one puts $\xi(t) \equiv \xi(t_0)$, and with the probability $1-r$, the "white noise" is simulated with the correlation function determined by relation (1.3) for $r = 0$.

4. Let

$$r(t, t') = e^{-\sigma|t-t'|}, \quad \sigma > 0.$$

For constructing a realization of such a process, the time axis is partitioned into some intervals, whose lengths are independent from each other and distributed equally with the density $\sigma e^{-\alpha x}$. In each interval, the value of realization is constant and is subordinate to the given one-dimensional law of distribution, values on different intervals being independent. The probability of coincidence of the values $\xi(t_1)$ and $\xi(t_2)$ is equal to the probability of the fact that the boundaries of a random partition in the interval (t_1, t_2) are absent, and, hence, it is equal to $\exp(-\sigma(t_2 - t_1))$, because of a random number of boundaries, got to (t_1, t_2), is distributed by the Poisson law with the parameter $\sigma(t_2 - t_1)$.

This paper also presents the generalization of the above-considered algorithm to the case of processes with the correlation function

$$r(\tau) = \int_0^\infty \exp(-x\tau) d\mu(x), \quad \tau = |t - t'|,$$

where μ is a non-negative measure in the interval $[0, \infty)$ such that $\int_0^\infty d\mu(t) \leq 1$.

In the general case, for the simulation of a random vector by the "repetition" method, in each realization the vector components may be partitioned into some groups of identical values and, moreover, for the fixed partition in different groups the values of the vector $\vec{\xi}$ are independent and subordinate to the one-dimensional distribution law. Hence follows the general principle of building the "repetition" method algorithm: for all possible variants Δ_m of the partition of the set of components into the above - indicated groups, it is necessary to determine probabilities p_m so that as a result one would obtain a random vector with the given correlation matrix. Hence, for the fixed partition Δ_m a random vector is simulated with the correlation matrix $R(\Delta_m) = (r_{ij}^m)$, formed of zeros and units so that $r_{ij} = 1$, if the components ξ_i and ξ_j belong to the same group. Thus, for determining the quantities p_m, it is necessary to consider the relation

$$R = \sum_{m=1}^{N} p_m R(\Delta_m), \tag{1.4}$$

where N is the full number of variants of the partition. For the sake of definiteness, one may consider that the enumeration of the partitions Δ_m is made according to decreasing the number of units in the matrices $R(\Delta_m)$ (for the same number of units the enumeration is arbitrary). Then $R(\Delta_1)$ is a matrix, consisting of units, and in $R(\Delta_N)$ these units are on the main diagonal. In this connection, relation (1.4) may be rewritten in the form of the following system of $n(n-1)/2$ algebraic eguations

$$p_1 + \sum_{m=2}^{N-1} p_m r_{ik}^{(m)} = r_{ik}, \quad i < k \tag{1,5}$$

and, in addition, $p_m \geq 0$ and $\sum_{m=1}^{N} p_m = 1$. If the solution with the considered properties exists, then one may simulate the vector $\vec{\xi} = (\xi_1, \ldots, \xi_n)^T$ by the "repetition" method, otherwise such a simulation is impossible. The formulated problem may be solved in the following way: by the standard techniques of the linear programming (for example, by the simplex-method) a non-negative solution $(p_1^*, p_2^*, \ldots, p_{N-1}^*)$ of system (1.5) with the minimal sum of components P^* is obtained; if $P^* \leq 1$, then one puts $p_N = 1 - P^*$, then a random vector $\vec{\xi}$ may be simulated by the "repetition" method, corresponding to randomized representation (1.4); if $P^* > 1$, then the "repetition" method is not applied. In detail, the method of solution of system (1.5) is described in [23].

3.2. Method for simulation of random processes and fields on point flows

As a generalization of the "repetition" method for random processes, according to which the time axis is partitioned into some random intervals, the method, suggested by G.A. Mikhailov, for simulation of random processes, where the Palm point flows are used as partition of the time axis or the spatial domain (see [24]), is considered.

Let us consider a point flow of the form [24]

$$\tau_k = \sum_{i=1}^{k} \eta_i, \quad k = 1, 2, \ldots; \quad \tau = 0, \tag{2.1}$$

where η_i are independent non-negative random variables with the distribution densities $f_i(x)$, $i = 1, 2, \ldots$ such that

$$f_i(x) = f(x) = F'(x), \quad k = 2, 3, \ldots; \quad f_1(x) = \mu^{-1}[1 - F(x)]. \tag{2.2}$$

It is assumed that $\mu = \int_0^\infty x f(x) dx < \infty$. It was shown in [39] that the flow τ_k is stationary, if and only if relations (2.2) are executed, moreover, as it was shown in [60], a probability $p_0(t) = P(k = 0; t)$, where k is the number of hits of points τ_k into the interval $[0, t]$, is connected with $f(x)$ and $f_1(x)$

by the relations

$$f(x) = -p_0''(x)/p_0'(0), \quad f_1(x) = -p_0'(x), \quad x \geq 0, \tag{2.3}$$

moreover,

$$p_0(t) = \mu^{-1} \int_t^\infty [1 - F(x)]dx. \tag{2.4}$$

From (2.3), (2.4) the following properties of $p_0(t)$, $t \geq 0$ take place

$$p_0''(t) \geq 0, \quad |p_0'(0)| < \infty, \quad p_0(0) = 1, \quad p_0(\infty) = 0, \quad p_0'(t) \leq 0. \tag{2.5}$$

Thus, if the non-negative function $P(k = 0; t) = p_0(t)$ with properties (2.5) is given, then functions (3) are the distribution densities for which relations (2.2) are executed.

The algorithm for simulation of the random process $\xi(t)$, $0 < t < T$ with the given one-dimensional distribution function $F_\xi(x)$ reduces to the following transformations:

(1a) Palm flow τ_k, $k = 1, 2, \ldots$; $\tau_0 = 0$ is simulated until the first outlet of a point of the flow from $[0, T]$.

(2a) In each interval (τ_{i-1}, τ_i), $i = 1, 2, \ldots$ one puts $\xi(t) \equiv \xi_i$, where ξ_i are independent random variables with the distribution function $F_\xi(x)$.

The process $\xi(t)$ is stationary and its normalized correlation function $K(t, t') = K(t)$ is coinsided with $p_0(t)$. Really

$$K(t, t') = \begin{array}{l} [M\xi(t')\xi(t' + t) - M\xi(t')M\xi(t' + t)] \,/\, \left(\sqrt{D\xi(t')D\xi(t' + t)}\right) = \\ = \frac{p_0(t)M\xi^2 + [1 - p_0(t)](M\xi)^2 - (M\xi)^2}{M\xi^2 - (M\xi)^2} = p_0(t) = K(t). \end{array},$$

It is assumed that $M\xi^2 < \infty$. Thus, a stationary process with a given one-dimensional distribution function $F_\xi(x)$ and with a convex normalized correlation function $K(t) = p_0(t)$ is built.

The realization of Palm flow τ_k may be constructed with the help of the method of "inverse distribution function" by the following formulas

$$K(\eta_1) = \alpha_1, \quad K'(\eta_i) = a_i K'(0), \quad i = 2, 3, \ldots,$$

where α_i, $i = 1, 2, \ldots$ are independent random variables uniformly distributed in the interval $(0, 1)$.

As an example the case of a absolutely monotonic function $K(t)$ in the form

$$K(t) = \int_0^\infty e^{-\lambda t}\mu(d\lambda), \tag{2.6}$$

was considered [108], where μ is a probability measure on $(0, \infty)$ such, that

$$P(\lambda = 0) = 0 \quad \text{and} \quad \int_0^\infty \lambda\mu(d\lambda) < +\infty.$$

In this case conditions (2.5) are executed and densities $f(x)$ and $f_1(x)$ have the following form

$$f(x) = \int_0^\infty \lambda^2 e^{-\lambda x}\mu(d\lambda) \left[\int_0^\infty \lambda\mu(d\lambda)\right]^{-1}, \quad f_1(x) = \int_0^\infty \lambda e^{-\lambda x}\mu(d\lambda).$$

If the measures μ and $\lambda\mu$ are conveniently simulated, then it is worth while to apply the superposition method for this purpose [25]. Note, that there exits a randomized method for simulation of a process with an absolutely monotonic correlation function $K(t)$ of the type (2. 6), where λ is a random parameter sampled with respect to μ. This process is not ergodic, while the process $\xi(t)$, simulated according to the rules $(1a)$ - $(2a)$, is ergodic and therefore more natural and important for applications.

3.3. Randomized models of non-Gaussian discrete processes

In this Section two models, suggested by A.S. Marchenko in paper [21], will be considered. These models are based on randomization of the autoregressive process and the process of moving average. Stationary series with an arbitrary one-dimensional probability distribution $F(x)$ and a special class of correlation functions (it includes only functions with non-negative values) may be built with the help of the following randomized procedure [21].

1. Let $\ldots\xi_{-1}, \xi_0, \xi_1, \ldots$ be independent and distributed according to $F(x)$, and $\gamma_t^{(0)}, \gamma_t^{(1)}, \ldots$ be a series of random variables, independent of ξ_t, $t = \ldots -1, 0, 1$ with the values 0 and 1 and, moreover, among all $\gamma_t^{(0)}, \gamma_t^{(1)}, \ldots$

only one variable is equal to unity. This series forms a system $F_t(\infty)$ with a countable set of states, which randomly transit from one to another for the time t. If

$$P\left(\gamma_t^{(k)} = 1,\ \gamma_{t+h}^{(k+h)} = 1\right)$$

is independent of t, then the series

$$\xi_t = \sum_{k=0}^{\infty} \gamma_t^{(k)} \xi_{t-k}, \quad t = 1, 2, \ldots \tag{3.1}$$

is stationary with the one-dimensional distribution $F(x)$ and the correlation function

$$\rho_h = \sum_{k=0}^{\infty} P\left(\gamma_t^{(k)} = 1,\ \gamma_{t+h}^{(k+h)} = 1\right), \quad h = 0, 1, \ldots\ . \tag{3.2}$$

In particular, if $\gamma_t^{(k)}$ and $\gamma_s^{(l)}$ for $t \neq s$ are independent and $P\left(\gamma_t^{(k)} = 1\right) = g_k$, then

$$\rho_0 = 1, \quad \rho_h = \sum_{k=0}^{\infty} g_k g_{k+h}, \quad h = 1, 2, \ldots, \tag{3.3}$$

and, in addition, $\sum_{k=0}^{\infty} \rho_k \leq 1$. This class consists of non-negative, decreasing functions, the decrease not necessarily being monotonic. If the system $F_t(\infty)$ is a homogeneous Markov chain with the probabilities $P\left(\gamma_t^{(k)} = 1\right) = g_k$ and the matrix G of one step transition probabilities, then ρ_h in this case is a combination of exponents λ_i^h, where λ_i are eigenvalues of the matrix G (here $\ldots \leq |\lambda_1| \leq |\lambda_0| \leq 1$).

In [87] some other classes of processes ξ_t are presented and, also, a randomized autoregressive model of the p-th order is considered.

2. Let $\zeta_{-p+1}, \ldots, \zeta_{-1}, \zeta_0, \zeta_1, \ldots$ be independent and distributed according to $F(x)$, and $\beta_t^{(1)}, \ldots, \beta_t^{(p+1)}$ be an independent of ζ_t, $t = \ldots -1, 0, 1, \ldots$ series of random variables, among which only one of them being equal to unit, and all the rest being zeros. Let, then $P\left(\beta_t^{(k)} = 1\right) = b_k$, $\sum_{k=1}^{p+1} b_k = 1$, $b_{p+1} > 0$, $\beta_t^{(k)}$ and $\beta_s^{(l)}$ for $t \neq s$ be independent. Then the series

$$\xi_t = \beta_t^{(1)} \xi_{t-1} + \ldots + \beta_t^{p} \xi_{t-p} + \beta_t^{p+1} \zeta_t, \quad t = 1, 2, \ldots \tag{3.4}$$

(here $\xi_i = \zeta_i$ for $i = 0, -1, \ldots, -p+1$) has the distribution $F(x)$ and is asymptotically stationary with respect to t with a correlation function of the form

$$\rho_h = \vec{e}_1^T B^h \vec{c}, \quad h = 0, 1, \ldots,$$

where $\vec{e}_1^T = (1, 0, \ldots, 0)$,

$$B = \left\| \begin{array}{cc} \vec{b}1^T & b_p \\ I & \vec{0} \end{array} \right\|,$$

$\vec{b}^T = (b_1, \ldots, b_{p-1})$, I is the identity matrix, $\vec{0}$ is zero vector-column, $\vec{c}$ is a constant vector depending on $b_1, b_2, \ldots, b_{p+1}$. For $h \geq p$ the correlation function of a series (3.4) satisfies the Jule - Walker equation

$$\rho_h = b_1\rho_{h-1} + \ldots + b_p\rho_{h-p}.$$

In particular, $\rho_h = b_1^h$ for $p = 1$.

3.4. Combined models of non-Gaussian process and fields

In this Section, a simple modification [10,27] of the methods for simulation of random processes and fields, considered in the previous two Sections, as well as methods for simulation of discrete processes and fields in combination with the methods based on point flows will be discussed. Similar processes where considered in [12].

As an example consider a simple case of such a combination. Let us consider the following procedure of constructing a random process:

($1a$) In the interval $[0, T]$ the grid points $t_1 = 0$, $t_2, \ldots, t_n = T$ are arbitrarily fixed. At these points a discrete random sequence $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ with the given one-dimensional distributions $F_{\xi(t_i)}(x)$ and a covariance matrix $K(t_i, t_j)$, $i, j = 1, \ldots, n$ is simulated (the matrix $K(t_i, t_j)$ can be either Toeplitz, or non-Toeplitz).

($2a$) In every interval $(t_1, t_2), (t_2, t_3), \ldots, (t_{n-1}, t_n)$ a random point x_i is sampled with respect to the probability density $f_i(x)$ $x \in (t_i, t_{i+1})$. These points form a random point flow and are random boundaries of the intervals $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-2}, x_{n-1})$, inside which the points $t_2, \ldots, t_{n-1}$ are contained. The intervals (t_1, x_1) and (x_{n-1}, t_n) are adjoining to boundaries of the interval $[0, T]$.

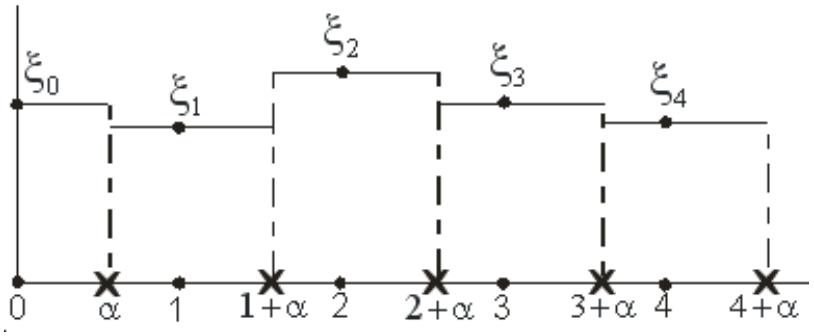


Figure 1: Piesewise constant random process

(3a) For every interval (x_i, x_{i+1}) we take $\xi(t) = \xi(t_i)$, $i = 2, \ldots, n-1$. For the intervals (t_1, x_1) and (x_{n-1}, t_n) we take $\xi(t) \equiv \xi(t_1)$ and $\xi(t) \equiv \xi(t_n)$, respectively.

Covariance function

$$K(t, t') = M\xi(t)\ \xi(t') - M\xi(t) M\xi(t')$$

of the process $\xi(t)$, $t \in [0, T]$ for the values $t = t_i$ and $t' = t_j$ coincides with the given covariance matrix $K(t_i, t_j)$ for an arbitrary probability density of random variables x_i in the intervals (t_i, t_{i+1}), $i, j = 1, \ldots, n$.

For values $t \neq t_i$ a one-dimensional distribution $F_{\xi(t)}(x)$ in the general case depends on the distributions $F_{\xi(t_i)}(x)$ and $f_i(x)$. In particular, if $F_{\xi(t_i)}(x) \equiv F(x)$, then $F_{\xi(t)}(x) \equiv F(x)$. The covariance function $K(t, t')$ for $t \neq t_i$, $t' \neq t_j$ depends on two-dimensional distributions of the corresponding pair of random variables x_i and x_j and, also, on the covariance matrix $K(t_i, t_j)$.

As an example consider the following process $\xi(t)$. Let

$$t_i = i, \quad F_{\xi(t_i)}(x) = F(x), \quad x_i = i + \alpha\varepsilon, \quad i, j = 1, 2, \ldots, n, \tag{4.1}$$

where α is a random variable, uniformly distributed in the interval $(0, 1)$. Construct a realization of a process $\xi(t)$ according to the above-considered procedure (1a) - (3a). For simplicity we take $M\xi(t_i) = 0$. Then the covariance between values of the process $\xi(t)$ at the points $t = i+\tau$ and $t' = j+\tau'$, where $\tau' > \tau$ and $\tau, \tau' \in [0, 1]$, has the following form

$$M\xi(t)\ \xi(t') = K(t, t') = (1 - t')\ \sigma_i^2 + (t' - t)\ K(i, j) + t\sigma_j^2. \tag{4.2}$$

Here $\sigma_i^2 = M\xi_2^2$, $K(i,j)$ is a given covariance matrix of the sequence ξ_i.

If $t' = t + h$, $\sigma_i^2 = \sigma_j^2 = 1$, then

$$K[i+\tau, (j-1)+\tau+h] = K[i,(j-1)+h] = 1 - hK(i,j)h$$

In the given case, a covariance function is independent of t, hence it can be used for approximation of covariance matrices both of the Toeplitz and non-Toeplitz types. For example, if α is a random variable distributed in the interval $(0,1)$ with the density $f(x) = 2x$, then the covariance function $K(t,t')$ depends, also, on τ:

$$K(t,t') = K[i,(j-1)+t+h] = 1 + (2th - h^2)\,(1 - K(i,j)). \qquad (4.3)$$

The selection of random points x_i in the intervals (t_i, t_{i+1}) can be realized by different methods, dependently or independently, with the same distribution density or with different ones, etc. All these transformations determine a family of covariance functions, approximating a given covariance matrix $K(t_i,t_j)$ and coinciding with it at the grid points.

Consider one interesting particular case [129]. Let, as in (4.1), $t_i = i$, $x_i = i+\alpha$, $i = 1,2,\ldots$, where α, be a random uniformly distributed variable in the interval $(0,1)$, and ξ_i be a stationary random discrete process with a correlation function $K(i,j) = K(|i-j|) = R_k$, $k = 0,1,\ldots$, $R_0 = 1$. According to the procedure $(1a)$ - $(3a)$ and based on the above-mentioned conditions (1), construct a random process $\xi(t)$ in the interval $[0,\infty)$ and write down the correlation function of the constructed process. Consider the interval $(i, i+1)$. The correlation coefficient between values of the process $\xi(t)$ at the points t and $t' \in (i, i+1)$ has the form

$$R(t,t') = (1-t') + (t'-t)R_1 + t.$$

Take $t' = t + h$, $h \in (0,1)$. Then

$$R(t,t') = (1-h) + hR_1.$$

Since the variable α is the same for all intervals (note, that the points x_i form a regular point flow), then the correlation coefficient between values of the process at the points $t \in (i, i+1)$ and $t' \in (i+k, i+k+1)$ is equal to

$$R(t,t') = (1-h)R_k + hR_{k+1}, \quad h \in (0,1).$$

Take $t' = t + \tau$, where $\tau = k + h$, $k = \text{entier}\,(\tau)$. Then

$$R(t,t') = R(t,t+\tau) = (1+k-\tau)R_k + (\tau-k)R_{k+1} = R(\tau), \quad k = \text{entier } (\tau). \tag{4.4}$$

The constructed process $\xi(t)$ is a stationary one. The correlation function of this process is a non-negative definite, piecewise-linear function, which coincides with the given values of the correlation function R_k for $k = 0, 1, 2 \dots$.

Thus, the statement, for the first time formulated by L. Shepp (see [43]), is followed from $(1a)$ - $(3a)$ and (4): if a function of a discrete argument $R(k) = R_k$, $R_0 = 1$, $k = 0, 1, 2 \dots$ is a non-negative definite function, then the function $R(\tau) = (1 + k - \tau)R_k + (\tau - k)R_{k+1}$, $k =$ entier (τ) is a non-negative definite one.

By analogy it is shown that this property is valid also for the processes, whose argument values are changing in bounded intervals.

The considered property in some cases can be used in applications for a piecewise-linear approximation of sampling correlation functions. Note, that in terms of the concrete considered technique it is impossible to find another approximation of a discrete correlation function and in doing so not to infringe stationarity of the process, because the unique factor, determining the behaviour of correlation dependence is a distribution of a random variable α in the interval (0,1). It is easy to show that a sample of a variable α with a non-uniform distribution density results to non-stationarity of such a process.

The spectral density of the process $\xi(t)$ $t \in [0, \infty)$ with the correlation function (4.4) has the following form

$$f(\lambda) = 2g(\lambda)\tilde{f}_1(\lambda), \tag{4.5}$$

where

$$g(\lambda) = \frac{1}{\pi} \frac{1 - \cos \lambda}{\lambda^2} = \frac{1}{2\pi} \left[\frac{\sin \frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)} \right]^2$$

is the spectral density, correponding to the process with the triangular correlation function

$$R(\tau) = \sum_{m=-\infty}^{\infty} K(\tau - m)R_m, \quad K(\tau) = \begin{cases} 1 - |\tau|, & |\tau| < 1, \\ 0, & |\tau| \geq 1. \end{cases}$$

and $\tilde{f}_1(\lambda)$ is a periodic continuation of the spectral density $f(\lambda)$ in the form

$$\tilde{f}_1(\lambda) = \left(1 + 2\sum_{k=1}^{\infty} R_k \cos \lambda k\right) \tag{4.6}$$

Relation (4.5) proves the existence of the piecewise-linear approximation of the correlation function R_k, $k = 0, 1, 2, \ldots$ under condition of the uniform convergence of series (4.6), i.e., if

$$\sum_{k=1}^{\infty} |R_k| < \infty$$

.

In conclusion of this Section consider a generalization of this method for the simulation of non-homogeneous non-Gaussian random fields $\xi(x)$, determined at an arbitrary point $x \in R^m$ of the domain D, with a covariance function $K(x, x')$, coinciding with a given covariance matrix $K(x_i, x_j)$ at fixed points of the domain. According to the method, described in [10], consider the following technique of construction of the field on regular grid:

(1b) On each coordinate axis in R^m regular grid points $r_k^{(i)}$, $i = 1, \ldots, m$, $k = 1, \ldots, p$, such that $r_{k+1}^{(i)} - r_k^{(i)} = \Delta r$, determining the regular grid $\{r_n\}$, $n = 1, \ldots, p^m$, are sampled. At these grid points a realization of a discrete random field $\xi(r_n)$ with a given distribution function $F_\xi(x)$ and covariance matrix $R(r_q, r_l)$, $q, l = 1, \ldots, p^m$, is constructed.

(2b) In each interval $(r_k^{(i)}, r_{k+1}^{(i)})$ a random point $x_k^{(i)}$ is sampled. In this case $r_{k+1}^{(i)} \in \left(x_k^{(i)}, x_{k+1}^{(i)}\right)$.

(3b) In each k cuboid, with the vertex formed by adjacent points of grid with coordinates $x_k^{(i)}$, $x_{k+1}^{(i)}$, $i = 1, \ldots, m$, one takes

$$\xi(r) \equiv \xi(r_k), \quad r_k = \left(r_k^{(1)}, \ldots, r_k^{(m)}\right)^T.$$

In particular, if $m = 2$, $\Delta r = 1$, $r_q^{(1)} = q$, $r_l^{(2)} = l$, $x_q^{(1)} = q + \alpha_1$, $x_l^{(2)} = l + \alpha_2$, $x, y, x', y' \in [0, 1]$, $q, l = 1, \ldots, p$, where α_1 and α_2 are independent random values, uniformly distributed in the interval $(0, 1)$, then

$$K(q_1 + x, l_1 + y, q_2 + x', l_2 + y') = (1 - x')(1 - y')\sigma^2_{q_1 l_1} +$$

$$+ y(1 - x')\sigma_{q_1 l_2} + +(1 - y')x\sigma_{q_2 l_1} + xy\sigma_{q_2 l_2} + (1 - x')(y' - y)K(q_1, l_1; q_1, l_2) +$$

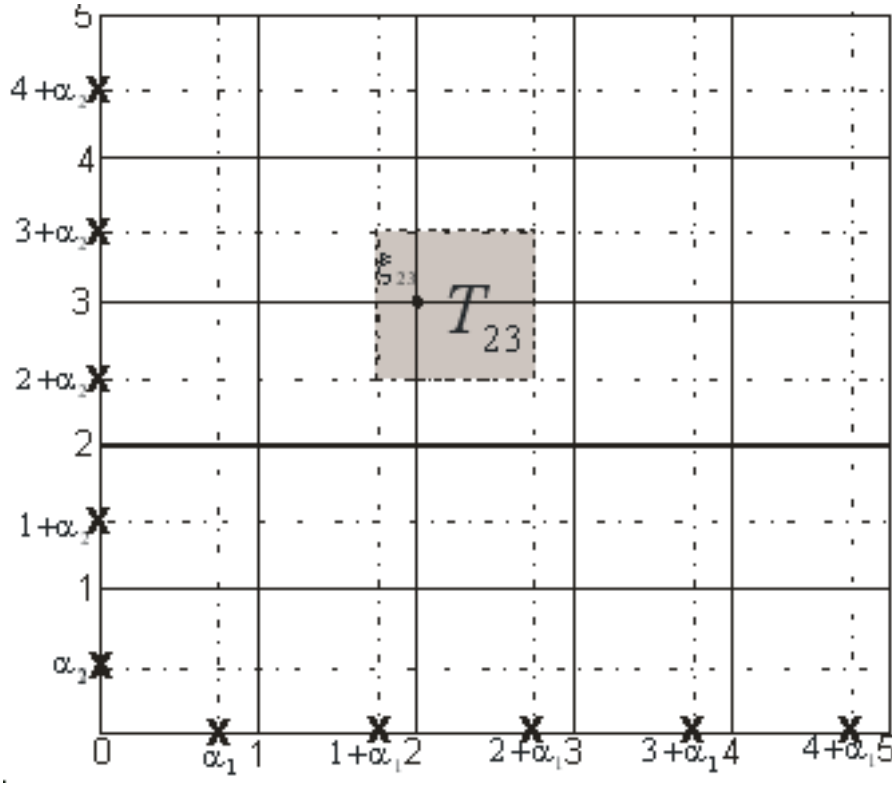


Figure 2: Simulation of two dimensional fields

$$+(1-y')(x'-x)K(q_1,l_1;q_2,l_1)+(y'-y)(x'-x)K(q_1,l_1;q_2,l_2)+$$
$$+y(x'-x)K(q_1,l_2;q_2,l_2)+x(y'-y)K(q_2,l_1;q_2,l_2)$$
.

By analogy with a one-dimensional case it may be shown that if $\xi(r_n)$, $n=0,1,\ldots$, $r_n\in\vec{R}^2$, be a homogeneous field with the correlation function $R(r_m,r_n)=R_{mn}$, then the field $\xi(r)$ constructed according to $(1b)$ - $(3b)$ is homogeneous one and its correlation function at the points $r_0=(x_0,y_0)^T$ and $r=(x_0+x,y_0+y)^T$ has the form

$$R(x_0,y_0;x_0+x,y_0+y)=K(x-m,y-m)R_{mn}+$$
$$+K(x-m-1,y-n)R_{m+1,n}+K(x-m,y-m-1)R_{m,n+1}+$$
$$+K(x-m-1,y-n-1)R_{m+1,n+1}=R(x,y)$$

where $m=$ entier (x), $n=$ entier (y), and

$$K(x,y)=\begin{cases}1-|x|-|y|+|x|\,|y|,|x|<1,\ |y|<1\\ 0,|x|\geq 1,\ |y|\geq 1.\end{cases}$$

The correlation function $K(x,y)$ is equal to zero outside of the interval $|x|<1$, $|y|<1$, therefore it may be presented in the form

$$R(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}K(x-m,y-n)R_{mn}.$$

Note some properties of the considered field $\xi(r)$.

1) Provided y is fixed we have the case of the scalar $(1a)$ - $(3a)$.

2) If $x=y$ then

$$R(x,y)=R(x)=\begin{cases}(1-|x|)^2,|x|<\sqrt{2}\\ 0,\qquad |x|\geq\sqrt{2}.\end{cases}$$

If a grid is irregular, one of simple algorithms can be constructed on the basis of the algorithm of simulation of homogeneous isotropic field [108] (see Section 3.4). Consider a case for $m=2$. Let x_i be $i=1,\ldots,p$ be coordinates of fixed points in the domain D in R^2:

$(1c)$ At the points $r_1,\ldots,r_p$, a realization of a discrete field $\xi(r_i)$ with a one-dimensional distribution $F_\xi(x)$ and a covariance matrix $K(r_i,r_j)$ $i,j=1,\ldots,p$ is simulated.

(2c) On the line z along an isotropic direction ω the projections of the poins r_i, determining the set of intervals (z_i, z_{i+1}), are found. In each interval a random point x_i is sampled, so that $z_{i+1} \in (x_i, x_{i+1})$.

(3c) At the points x_i, $i = 1, \ldots, p-1$ the lines, perpendicular to ω, are constructed. In each subdomain D_i of the domain D, bounded by the lines passing through x_i and x_{i+1}, one takes $\xi(r) \equiv \xi(r_i)$.

Unlike the method, considered in the previous Section, a covariance matrix of the thus conctructed field depends on all elements of the matrix $K(r_i, r_j)$. Note, that the considered methods are easy to generalize to a case of vector and spatial-time random fields. In Chapter 6, they will be used for the simulation of non-homogeneous non-Gaussian fields of some weather elements.

3.5. Piecewise-constant filling of a periodically correlated process

Consider the problem of piecewise-linear stochastic filling of a periodically correlated process $\xi(t_i)$ from nodes $t_i$and t_{i+1} to an arbitrary point of the segment $[t_i, t_{i+1}]$, $i = \ldots, -1, 0, 1, \ldots$. For simplicity sake, we present all the results for the case to $t_0 = 0$, $\Delta t = t_{i+1} - t_i = 1$, nd $T = p\Delta t = p$. The passage to arbitrary t_i and Δt is performed by corresponding shifts and stretching of the grid. Let us show that the piecewise-linear filling preserves the original structure of the process. Problems of this type were considered in [5, 10].

Let α be a random value distributed over the segment [0, 1] with a distribution function $F(x)$. Construct a random process $\eta(t)$ of a continuous argument from the process $\xi(t_i) = \xi(i)$ in the following way [5]:

$$\eta(t) = \begin{cases} \xi(i), & t \le \alpha + i \\ \xi(i+1), & t > \alpha + i \end{cases}, \quad t \in [i,\ i+1],\ \ i = \ldots, -1, 0, 1, \ldots. \tag{5.1}$$

Let us show that the mean of the process $\eta(t)$ satisfies the first condition from (5.1). Averaging $\eta(t)$, we get

$$E\eta(t) = \begin{cases} E\xi(i), & t \le \alpha + i \\ E\xi(i+1), & t > \alpha + i \end{cases}, \quad t \in [i,\ i+1],\ \ i = \ldots, -1, 0, 1, \ldots.$$

Then

$$E\eta(t+T) = \begin{cases} E\xi(i+T), & t+T \le \alpha + i + T \\ E\xi(i+1+T), & t+T > \alpha + i + T \end{cases},$$

where $t+T \in [i+T,\ i+1+T)\,,\ \ i=\ldots,-1,0,1,\ldots$.

This obviously implies that the first relation from (1.1) holds for the process $\eta(t)$.

$$D\eta(t) = E\eta^2(t) - (E\eta(t))^2. \tag{5.2}$$

It is easy to see that

$$E\eta^2(t) = F(\{t\})E\xi^2(i+1) + (1-F(\{t\}))E\xi^2(i),$$

where $t \in [i,\ i+1]\,,\ \ i=\ldots,-1,0,1,\ldots.$, and $\{t\}$ is the fractional part of the number t. Then

$$E\eta^2(t+T) = F(\{t\})E\xi^2(i+1+T) + (1-F(\{t\}))E\xi^2(i+T), \tag{5.3}$$

where $\{t+T\} = \{t\}$.

Taking into account the two first relations from (5.1) for the process $\xi(t_i)$, we get $E\xi^2(t_i+T) = E\xi^2(t_i)$ and hence $E\xi^2(t_{i+1}+T) = E\xi^2(t_{i+1})$. Taking into account these equalities, relation (5.2) implies $E\eta^2(t+T) = E\eta^2(t)$ and finally

$$D\eta(t+T) = D\eta(t).$$

In order to calculate the correlation function $R_\eta(t,\ t+h)$ of the process $\eta(t)$, we have to consider two variants:

1. $\{t\} \le \{t+h\}, \quad t,h \in (-\infty,\infty)$

$$\begin{gathered} R_\eta(t,\ t+h) = \\ = F(\{t\})R_\xi([t]+1,\ [t+h]+1) + (F(\{t+h\}) - F(\{t\}))R_\xi([t],\ [t+h]+1)+ \\ +(1-F(\{t+h\}))R_\xi([t],\ [t+h]) \end{gathered}$$

where $[x]$ denotes the integer part of a real number.

2. $\{t\} > \{t+h\}, \quad t,h \in (-\infty,\infty)$

$$\begin{gathered} R_\eta(t,\ t+h) = \\ = F(\{t+h\})R_\xi([t]+1,\ [t+h]+1) + (F(\{t\}) - F(\{t+h\}))R_\xi([t]+1,\ [t+h])+ \\ +(1-F(\{t\}))R_\xi([t],\ [t+h]) \end{gathered}$$

Since T is integer, taking into account (1.1), for $\{t\} \le \{t+h\}, \quad t,h \in (-\infty,\infty)$we have

$$\begin{gathered} R_\eta(t+T,\ t+h+T) = \\ = F(\{t+T\})R_\xi([t+T]+1,\ [t+h+T]+1)+ \\ +(F(\{t+h+T\}) - F(\{t+T\}))R_\xi([t+T],\ [t+h+T]+1)+ \\ +(1-F(\{t+h+T\}))R_\xi([t+T],\ [t+h+T]+1)+ \\ = F(\{t\})R_\xi([t]+1,\ [t+h]+1) + (F(\{t+h\}) - F(\{t\}))R_\xi([t],\ [t+h]+1)+ \\ +(1-F(\{t+h\}))R_\xi([t],\ [t+h]) = R_\eta(t,\ t+h) \end{gathered}$$

Similarly, for $\{t\} > \{t+h\}, \quad t, h \in (-\infty, \infty)$, we have

$$R_\eta(t+T,\ t+h+T) = R_\eta(t,\ t+h).$$

Thus, process (5.1) is periodically correlated.

Note that the inequalities $\{t\} \le \{t+h\},\ \{t\} > \{t+h\},\ t, h \in (-\infty, \infty)$ are equivalent to $F(\{t+h\}) - F(\{t\}) \ge 0,\ F(\{t+h\}) - F(\{t\}) < 0, t, h \in (-\infty, \infty)$, therefore, if the original process $\xi(t_i) = \xi(i)$ is stationary, i.e.,

$$R_\xi(i+n,\ i+m) = R_\xi(m-n)$$

then $R_\xi([t]+1,\ [t+h]+1) = R_\xi([t],\ [t+h])$ and the correlation function $R_\eta(t,\ t+h)$ of the process $\eta(t)$ takes the form

$$\begin{gathered} R_\eta(t,\ t+h) = \\ = (1 - |F(\{t+h\}) - F(\{t\})|) R_\xi([t],\ [t+h]) + \\ + |F(\{t+h\}) - F(\{t\})|\, R_\xi([t],\ [t+h]+1). \end{gathered}$$

If a random variableα is uniformly distributed in the interval (0,1), then

$$\begin{gathered} R_\eta(t,\ t+h) = \\ = (1 - |\{t+h\} - \{t\}|) R_\xi([t],\ [t+h]) + |\{t+h\} - \{t\}|\, R_\xi([t],\ [t+h]+1) = \\ = (1 - |\{h\}|) R_\xi([t],\ [t+h]) + |\{h\}|\, R_\xi([t],\ [t+h]+1) \end{gathered}$$

Assume $[t+h] - [t] = k$ and $\{h\} = \tau$, where $|\tau| \le 1$. Then in each interval $t \in [k,\ k+1)\,,\ \ k = \ldots, -1, 0, 1, \ldots$ we have

$$R_\eta(t,\ t+h) = R_\eta(h) = (1 - |\tau|) R_\xi(k) + |\tau|\, R_\xi(k+1)$$

or, which is equivalent (see [10]),

$$\begin{gathered} R_\eta(h) = \sum_{k=-\infty}^{\infty} K(h-k) R_\xi(k), \\ K(x) = \begin{cases} 1 - |x|\,, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} . \end{gathered}$$

3.6. Numerical stochastic models of piecewise-linear random processes

For construction of probabilistic models of actual processes and fields, for example, meteorological or oceanographic processes, economic and price series, the approaches considered above are based on the usage of empirical

correlation functions and one-dimensional distributions. In the case when correlation connections are sufficiently weak, for example, in price series, it is expedient to utilize other approaches not connected with using correlation functions and one-dimensional distributions. One of such approaches is based on piecewise-linear approximation of random actual processes [36]. In this approach, parameters of the linear segments of an approximating function are estimated based on observation data and then the parameters are implemented according to the obtained estimates and a random piecewise-linear function is constructed. In some cases models of random processes with piecewise-linear trajectories are used in construction of climatic models of atmospheric processes and development of methods for long-term weather forecast.

In this paper we study some special processes related to the approach described above. In particular, we consider stepwise and ruled processes with piecewiseconstant and piecewise-linear trajectories generated by point flows.

3.6.1. Stepwise processes on Poisson flows

(a) Consider a sequence of independent positive random variables X_i with the density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, \, x \geq 0 \\ 0, \qquad x < 0 \end{cases} \tag{6.1}$$

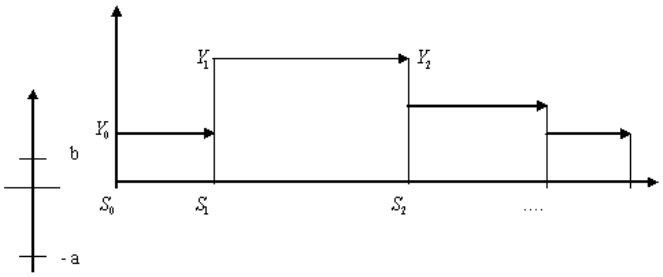
and the sequence of their sums $S_0 = 0$; $S_k = \sum\limits_{i=1}^{k} X_i$. For the sake of simplicity, consider the parameter $\lambda = 1$. Let $\alpha_i$be mutually independent random variables not dependent on X_i and uniformly distributed in the interval [-a ;b], a, b > 0 and

$$Y_k = \sum_{i=0}^{k} \alpha_i \tag{6.2}$$

The sequences $\{X_i\}$, $\{\alpha_i\}$ determine a stepwise random process on the positive semiaxis $t > 0$ (see Fig. 1):

$$Y(t) \equiv Y_k, \quad S_k \leq t < S_{k+1}, \quad k = 0, 1, \ldots \tag{6.3}$$

Let us obtain an expression for the mean value of the function $Y(t)$. Based on the total probability formula for mathematical expectations, write



down the expression for the function $E\eta\,(t)$ for any time t:

$$EY(t) = \sum_{k=0}^{\infty} P(B_k) E[Y(t)\,|B_k\,], \tag{6.4}$$

where the event

$$B_k\ = \{S_k\ < t,\ S_{k+1}\ > t\}$$

means that the random interval (ξ_k, ξ_{k+1}) covers the point t.

It is known (see, e.g., [6]) that the expressions for the probabilities $P(B_k) = P\,(S_k\ <\ t,\ S_{k+1}\ >\ t)$ have the form (the Poisson distribution)

$$P(B_k) = \frac{e^{-t}\,t^k}{k!}, \tag{6.5}$$

and (6.4) takes the form

$$E\eta(t) = \sum_{k=0}^{\infty} \frac{e^{-t}\,t^k}{k!} E \sum_{i=0}^{k} \alpha_i = \sum_{k=0}^{\infty} \frac{e^{-t}\,t^k}{k!} \frac{(b-a)}{2}(k+1) = \frac{(b-a)}{2}(t+1).$$

The graph of the function $E\eta(t)$ is presented in Fig. 2 for the parameter values =5, b = 2 (the upper line of the graph).

(b) Consider a stepwise random process on the positive semiaxis t > 0 taking the constant value Y_{k+1} in the interval (S_k, S_{k+1}) (i.e., the value at the right point S_{k+1} see Fig. 3)

$$Y(t) \equiv Y_{k+1}, \quad S_k \le t < S_{k+1}, \quad k = 0, 1, \ldots \tag{6.6}$$

Here and elsewhere the values S_k andY_k have the same meaning as in item (a).

Figure 3: Graph of the function $EY(t)$ for the cases: (a) upper line, (b) lower line, and (c) for =5, b = 2.

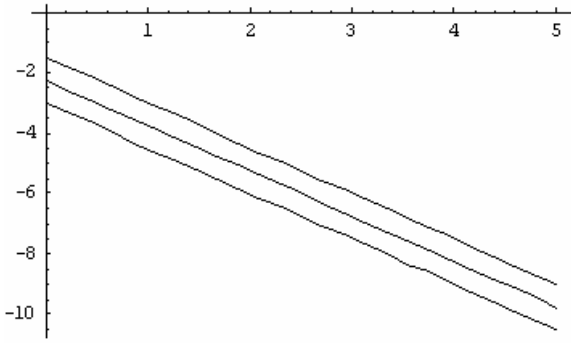
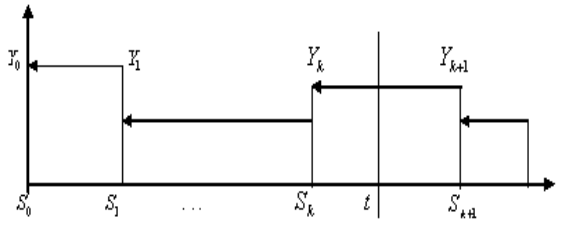


Figure 4:



In this case $E\eta(t)$ takes the form

$$EY(t) = \sum_{k=0}^{\infty} \frac{e^{-t}\, t^k}{k!} E \sum_{i=0}^{k+1} \alpha_i = \sum_{k=0}^{\infty} \frac{e^{-t}\, t^k}{k!} \frac{(b-a)}{2}(k+2) = \frac{(b-a)}{2}(t+2)$$

The graph of the function $EY(t)$ is presented in Fig. 2 (the lower line of the graph).

(c) Now consider a stepwise random process on the positive semiaxis $t > 0$ taking the constant value $(Y_k + Y_{k+1})/2$ in the interval (S_k, S_{k+1})

$$Y(t) \equiv \frac{Y_k + Y_{k+1}}{2}, \quad S_k \le t < S_{k+1}, \quad k = 0, 1, \ldots \tag{6.7}$$

For this variant of the process, $E\eta(t)$ takes the form

$$Y(t) = \sum_{k=0}^{\infty} \frac{e^{-t}\, t^k}{k!} E(\sum_{i=0}^{k} \alpha_i + \\ + \frac{\alpha_{k+1}}{2}) = \sum_{k=0}^{\infty} \frac{e^{-t}\, t^k}{k!} \frac{(b-a)}{2}(k+1.5) = \frac{(b-a)}{2}(t+1.5).$$

The graph of the function $E\eta(t)$ is presented in Fig. 2 (the middle line of the graph).

(d) Consider a discrete random process ω_k, $k = 1, 2, \ldots$, independently taking two values 0 or 1 with the probabilities

$$P(\omega_k = 0) = p_0, \quad P(\omega_k = 1) = p_1 = 1 - p_0$$

and a stepwise random process related to the former one:

$$\eta(t) \equiv \eta_k, \quad S_k \le t < S_{k+1}, \quad k = 0, 1, \ldots \tag{6.8}$$

where $S_0 = 0$; $S_k = \sum_{i=1}^{k} X_i$, X_i independent positive random values with the densities

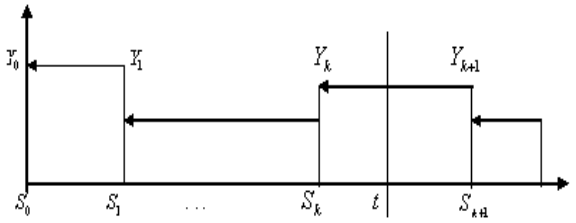
$$f^{(0)}(x) = e^{-\lambda_0 x}, \ for\ \omega_i = 0$$
$$f^{(1)}(x) = e^{-\lambda_1 x}, \ for\ \omega_i = 1$$

and η_k are mutually independent random variables not dependent on X_i and distributed with the distribution function $F_{\eta_k}(x)$ (see Fig. 4)

The process ω_k, $k = 1, 2, \ldots$, plays the controlling role in this model. The random value X_i may be also represented in the form

$$X_k = (1 - \omega_k) X_k^{(1)} + \omega_k X_k^{(2)}$$

Figure 5:



where $X_k^{(1)}$ and $X_k^{(2)}$ are mutually independent random values distributed with the densities $f^{(0)}(x)$ and $f^{(1)}(x)$, respectively. The distribution density of the random value X_i has the form

$$f(x) = p_0 f^{(0)}(x) + p_1 f^{(1)}(x) \tag{6.9}$$

Thus, the problem is reduced to the implementation of a random process on Palm point flows [7] where the distribution of the flow interval has form (6.9).

3.6.2. Ruled processes on Poisson flows

(a) Consider a ruled random process on the positive semiaxis $t > 0$ taking the following values in the interval (S_k, S_{k+1}):

$$\begin{aligned} Y(t) &= \tfrac{Y_{k+1}-Y_k}{S_{k+1}-S_k}\,(t - S_k) + Y_k = \\ &= \tfrac{\alpha_{k+1}}{S_{k+1}-S_k}\,(t - S_k) + \sum_{i=0}^{k} \alpha_i, \quad S_k \le t < S_{k+1}, \quad k = 0, 1, \ldots \end{aligned} \tag{6.10}$$

As in all previous cases, here $S_0 = 0$; $S_k = \sum\limits_{i=1}^{k} X_i$, X_i are independent positive random variables with density (6.1) for $\lambda = 1$, $Y_k = \sum\limits_{i=0}^{k} \alpha_i$, α_i mutually independent random variables not dependent on X_i and uniformly distributed in the interval $[-a\,;\,b]$, $a, b > 0$ (see Fig. 5).

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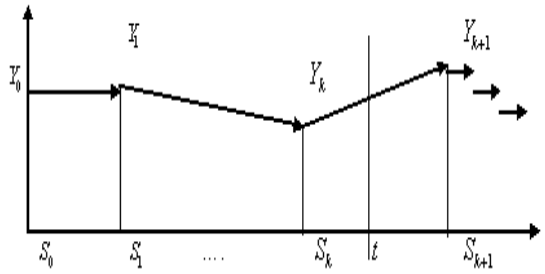
Based on total probability formula (6.4), we obtain the mathematical expectation for function (6.10). The probability of the event B_k, $k =$

Figure 6:



$0, 1, \ldots$, corresponding to the point t being covered by the interval (S_k, S_{k+1}) is determined by formula (6.5).

Using the notation

$$\frac{t - S_k}{S_{k+1} - S_k} = V_k(t)$$

the conditional mathematical expectation of the first summand in formula (6.10), provided the event B_k, $k = 0, 1, \ldots$, is realized, is determined by the expression

$$E(\alpha_{k+1} V_k(t) \,|B_k\,) = \int\limits_{-a}^{b} \int\limits_{0}^{t} \int\limits_{t}^{\infty} y_1 \frac{t - y_2}{y_3 - y_2} f_{\alpha_{k+1},\, S_k,\, S_{k+1}}(y_1,\, y_2,\, y_3 \,|B_k\,) dy_3 dy_2 dy_1$$

The expression for the joint density of the values α_{k+1} , S_k, S_{k+1} under the condition B_k, $k = 1, 2, \ldots$, has the form:

$$f_{\alpha_{k+1},\, S_k, S_{k+1}}(y_1,\, y_2,\, y_3 \,|B_k\,) = \frac{f_{\alpha_{k+1},\, S_k, S_{k+1}}(y_1, y_2, y_3)}{P(B_k)} =$$
$$= \frac{\frac{1}{b+a} \frac{y_2^{k-1}\, e^{-y_3}}{(k-1)!}}{\int\limits_{t}^{\infty} \int\limits_{0}^{t} \frac{y_2^{k-1}\, e^{-y_3}}{(k-1)!} dy_2 dy_3} = \frac{1}{b+a} \frac{\frac{y_2^{k-1}\, e^{-y_3}}{(k-1)!}}{\frac{t^k}{k!} e^{-t}} = \frac{1}{b+a} \frac{k\, y_2^{k-1}\, e^{t-y_3}}{t^k}.$$

Here

$$f_{\alpha_{k+1},\, S_k, S_{k+1}}(y_1,\, y_2,\, y_3) = \frac{1}{b+a} \frac{y_2^{k-1}\, e^{-y_3}}{(k-1)!} \tag{6.11}$$

represents the joint density of the values , ψ_{k+1} , ξ_k, ξ_{k+1}.

For $k = 0$ the random value $\alpha_{k+1}\frac{t- S_k}{S_{k+1}-S_k}$ takes the form $\frac{\alpha_1}{X_1}\,t$ The joint density of the values α_1 , X_1 under the condition $B_k = \{ S_1 > t\}$ equals

$$f_{\alpha_1,\, S_1}(y_1,\ y_3\,|B_0\,) = \frac{1}{b+a}\,\frac{e^{-y_3}}{e^{-t}}$$

Since by the construction of process (2,1) the value $\sum\limits_{i=0}^{k}\alpha_i$ depend on the values α_{k+1}, S_k, S_{k+1}, then, under the condition of realization of the event B_k, $k = 0, 1, 2, \ldots$ the conditional mathematical expectation of this value coincides with the unconditional one and is equal to $(k+1)\frac{b-a}{2}$

As a result, the total probability formula for mathematical expectations (6.4) for process (6.10) takes the form

$$\begin{gathered} EY(t) = \tfrac{t^0}{0!}e^{-t}\int\limits_{-a}^{b}\int\limits_{t}^{\infty}\tfrac{1}{b+a}\,\tfrac{y_1}{y_3}\,\tfrac{e^{-y_3}}{e^{-t}}dy_3dy_1 + \\ + \sum_{k=1}^{\infty}\tfrac{t^k}{k!}e^{-t}\int\limits_{-a}^{b}\int\limits_{0}^{t}\int\limits_{t}^{\infty}\tfrac{1}{b+a}\,\tfrac{y_1(t-y_2)}{y_3-y_2}\ \tfrac{k\,y_2^{k-1}\,e^{t-y_3}}{t^k}dy_3dy_2dy_1 + \\ + \sum_{k=0}^{\infty}\tfrac{t^k}{k!}e^{-t}(k+1)\tfrac{b-a}{2}. \end{gathered}$$

Denote the first two summands in this expression by $\frac{b-a}{2}C(t)$. After some transformations we get

$$EY(t) = \frac{b-a}{2}(\,C\,(t)\ + t + 1)$$

where $C(t)$ has the form

$$C(t) = \int\limits_{t}^{\infty}\frac{t}{y_3}e^{-y_3}dy_3 + \int\limits_{0}^{t}e^{y_2}(\int\limits_{t}^{\infty}\frac{(t-y_2)}{y_3-y_2}\,e^{-y_3}dy_3)dy_2. \tag{6.12}$$

The values $v_0 = \frac{t}{y_3}$ and $v_1 = \frac{t-y_2}{y_3-y_2}$ satisfy the inequalities

$$\begin{gathered} 0 \le v_0 \le 1, \\ 0 \le v_1 \le 1. \end{gathered}$$

If we assume $v_0 = 0$and $v_1 = 0$, then $C(t) = 0$, and if v_0 and v_1 are equal to 1, then

$$\begin{gathered} C(t) = \int\limits_{t}^{\infty}e^{-y_3}dy_3 + \int\limits_{0}^{t}e^{y_2}(\int\limits_{t}^{\infty}e^{-y_3}dy_3)dy_2 = \\ = e^{-t} + (e^t - 1)e^{-t} = 1. \end{gathered}$$

Thus,

$$0 \le C(t) \le 1 \tag{6.13}$$

Using the notation: $k = \nu(t)$ is the number of a random interval covering the point t such that $S_{\nu(t)} < t \le S_{\nu(t)+1}$ [2, 4], let us consider function (6.10) in the form

$$Y(t) = \frac{Y_{\nu(t)+2} - Y_{\nu(t)+1}}{S_{\nu(t)+1} - S_{\nu(t)}} (t - S_{\nu(t)}) + Y_{\nu(t)+1} =$$
$$= (Y_{\nu(t)+2} - Y_{\nu(t)+1}) \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} + Y_{\nu(t)+1},$$

where, in accordance to (6.2), $Y_{\nu(t)+2}$ and $Y_{\nu(t)+1}$ denote sums of $\nu(t)+2$and $\nu(t)+1$ independent random values uniformly distributed in the interval $[-a\,;\,b]\,,\ a,\ b > 0$

The random value

$$V_{\nu(t)} = \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}}$$

is determined in the interval (0,1), i.e.,

$$0 \le \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} \le 1$$

therefore,

$$0 \le EV_{\nu(t)} \le 1$$

This gives

$$E\eta(t) = \frac{b-a}{2} E \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} + EY_{\nu(t)+1} =$$
$$= \frac{b-a}{2}(E \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} + E\nu(t) + 1) = \frac{b-a}{2}(E \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} + t + 1). \tag{6.14}$$

Note that the following identity holds:

$$EV_{\nu(t)} = E \frac{t - S_{\nu(t)}}{S_{\nu(t)+1} - S_{\nu(t)}} = C(t) \tag{6.15}$$

where $C(t)$ has form (6.12). In order to determine the dependence of $EV_{\nu(t)}$ on time t this value has been calculated by direct implementation for various values of t. The corresponding estimates are presented in the second column of Table 1. The third column contains values of $C(t)$ calculated based on formula (6.12) by using direct numerical solution of the integrals contained

in this formula and represented as expansion into a series in powers of $1/t$. The table contains values of $C(t)$ calculated using 100 expansion terms, below we present an example of the expression for $C(t)$ with 6 expansion terms:

$$C(t) \approx e^{-t}\left(\frac{1}{2}e^{t} - \left[\frac{1}{t}\right]^{2} + 6\left[\frac{1}{t}\right]^{3} - \right. \\ \left. -36\left[\frac{1}{t}\right]^{4} + 240\left[\frac{1}{t}\right]^{5} - 1800\left[\frac{1}{t}\right]^{6}\right. . \tag{6.16}$$

Table 4:

t	Estimate of $M\chi_{\nu(t)}$ by statistical modelling method	Values of $M\chi_{\nu(t)}$ calculated with the help of expansion of $C(t)$ into a series in powers of $1/t$
0.0	0.0000	0.0000
0.25	0.3067	0.3070
0.5	0.3949	0.3950
1.0	0.4605	0.4612
2.0	.4924	0.4926
3.0	.4982	0.4983
4.0	.4999	0.4996
5.0	.5007	0.4999

Table 1 shows that this value tends to $1/2$ for $t \to \infty$ within the statistical error precision.

With the use of relations (6.13) and (6.15) it is easy to show that function (6.14) satisfies the following inequality

$$\frac{b-a}{2}(t+1) < |EY(t)| \leq \frac{b-a}{2}(t+2)$$

In conclusion, we note that the models considered here are easily generalized to the case when linear and stepwise segments alternate and this alternation is performed with the help of a stochastic control process.

The specific character of nonstationary processes consists in the fact that their trajectories are piecewise-constant and piecewise-linear functions. The character of the mean values of these processes as functions of time (or trends) is determined by the method of their construction and the distribution of intervals in a point flow. In the case of a ruled process on Poisson flows, the trends show asymptotically linear behavior with increasing time

t , but in the neighborhood of $t = 0$ they have nonlinear dependence on time. For stepwise processes on Poisson flows, the trends are linear on the whole positive semiaxis $t > 0$. The parameters of the corresponding linear functions are determined by the distribution parameters of the values X_i and α_i. In solving practical problems, it is often necessary to simulate numerically processes with nonlinear trends. One can consider the method based on the usage of nonstationary point flows as a possible technique for numerical construction of such processes.

Tasks

1.Let $\vec{\xi}_n = (\xi_1, \ldots, \xi_n)^T = (\vec{\xi}^{(1)T}, \vec{\xi}^{(2)T})^T$ is distributed $N(\mu_n,\ R_n)$- with $M\vec{\xi}_n = \vec{\mu}_n = (\vec{\mu}^{(1)T}, \vec{\mu}^{(2)T})^T$

$$R_n = M(\vec{\xi}_n - \vec{\mu}_n)(\vec{\xi}_n - \vec{\mu}_n)^T = \left\| \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right\|$$

a) Show that if $R_{12} = R_{21} = 0$, then $\vec{\xi}^{(1)}$ is distributed $N(\mu^{(1)},\ R_{11})$, $\vec{\xi}^{(2)}\ - N(\mu^{(2)},\ R_{22})$.

b) Prove that if $R_{12} \neq 0,\ R_{21} \neq 0$, then any vector of subvector $\vec{\xi}_n =$ is normally distributed with parameters determined by vector $\vec{\mu}_n$ and covariance matrix R_n.

2. In task 1. obtain expressions for the for the blocks of covariance matrix of the vector $\vec{\eta}_n = C\vec{\xi}_n$, where

$$C = \left\| \begin{array}{cc} I & R_{12}R_{22}^{-1} \\ 0 & I \end{array} \right\|.$$

3. Let C is a nonsingular square matrix. Show that if $\vec{\xi}_n$ is distributed as $N(\vec{\mu}_n,\ R_n)$, then the vector $\vec{\eta}_n = C\vec{\xi}_n$ is distributed as $N(C\vec{\mu},\ CR_nC^T)$.

4. Let $\vec{\xi}_n$ is distributed $N(\vec{\mu}_n,\ R_n)$. Show that the vector $\vec{\xi}_n + \vec{b}$ is distributed $N(\vec{\mu}_n + \vec{b},\ R_n)$, where $\vec{b}$ is a arbitrary vector.

5. Show that $R_n = M(\vec{\xi}_n - \vec{\mu}_n)(\vec{\xi}_n - \vec{\mu}_n)^T$ is a positive definite matrix.

6. let C is any rectangular matrix. $p \times n$. Show that if $\vec{\xi}_n$ is distributed as $N(\vec{\mu}_n,\ R_n)$, then the vector $\vec{\eta}_n = C\vec{\xi}_n$ is distributed as $N(C\vec{\mu},\ CR_nC^T)$.

7. In exercise 1. get the expression for the conditional normal density of the vector $\vec{\xi}^{(1)}$ provided that $\vec{\xi}^{(2)} = \vec{x}^{(2)}$.

8. Get the expression for the two-dimensional normal density. Get the expression for the conditional two-dimensionalconditional normal density

9. Using (1.2.9) get formulas for modelling of Gaussian vector $\vec{\xi}_{(n)} = (\xi_1, , \ldots, \xi_n)^T$ with zero mean and covariance matrix R_n of the form (1.2.10).

10. Get relations (1.2.9).

11. Let R is any block square matrix

$$R = \left\| \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right\|.$$

where $R_{11}$and R_{22} are square matrices of different dimensions and R_{11} is non-singular. Show that

$$\det(R) = \det(R_{11})\ \det(R_{22} - R_{21}R_{11}^{-1}R_{12}).$$

12. Show that in (1.2.13) the quantities $d_k^2 > 0, \quad k = 1, 2, ..., n$.

13. Prove equality $(B^{-1})^T = (B^T)^{-1}$.

14. To derive formulas for the calculation of matrix elements of matrix T_n in decomposition $R_n^{-1} = T_n^T T_n$ (1.2.18), if the elements of matrix R_n^{-1} are known.

15. Receive expressions for block matrix, inverse to the matrix

$$R_{k+1} = \left\| \begin{array}{cc} R_k & J_k\vec{r}_k \\ \vec{r}_k^T J_k & 1 \end{array} \right\|.$$

16. Receive the relations (1.3.6) for the system of equations

$$R_{k+1}\vec{b}[k+1] = \vec{r}_{k+1}.$$

17. Prove the equality

$$d_k^2 = (1 - b_k^2[k])d_{k-1}^2.$$

18. Prove the equality $\det(I - \vec{x}\vec{y}^T) = 1 - \vec{y}^T\vec{x}$.

19. Prove the equality $(I - \vec{x}\vec{y}^T)^{-1} = I + \frac{\vec{x}\vec{y}^T}{1-\vec{y}^T\vec{x}}$.

20. Using the equality from Task 18 and connections

$$V_k = I_k - J_k\vec{b}[k]\vec{r}_k^T \Big/ r_{k+1}, \quad W_k = I_k - \vec{b}[k]\vec{r}_k^T,$$
$$\tilde{V}_k = I_k - J_k\vec{\beta}[k]\vec{g}_k^T \Big/ g_{k+1}, \quad \tilde{W}_k = I_k - \vec{b}[k]\vec{g}_k^T,$$

and assuming $r_{k+1} \neq 0$ with the help (1.3.6) receive the relation

$$b_{k+1}[k+1] = \frac{\beta_{k+1}[k+1]}{1-\varepsilon} \frac{\det\left(V_k \tilde{V}_k^{-1}\right)}{\det\left(W_k \tilde{W}_k^{-1}\right)}.$$

21. Prove the equality

$$b_{k+1}[k+1] = \frac{\beta_{k+1}[k+1]/(1-\varepsilon) + \vec{r}_k^T J_k(\vec{\beta}[k] - \vec{b}[k])\Big/\delta_k^2}{\vec{r}_k^T[(1-\varepsilon)\vec{\beta}[k] - \vec{b}[k]]\Big/\delta_k^2 + 1}.$$

22. Prove the equality

$$\det(\Phi_m - \lambda I_m) =$$
$$= \det\left(\left\| \begin{array}{ccccc} b_1-\lambda & b_2 & \dots & b_{m-1} & b_m \\ 1 & -\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & \dots & 1 & -\lambda \end{array} \right\|\right)$$
$$= \lambda^m - b_1\lambda^{m-1} - \dots - b_m$$

23. Prove the equality

$$R_m - \Phi_m R_m \Phi_m^T = \left\| \begin{array}{cccc} d^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array} \right\|,$$

$$d^2 = 1 - r_m^T \vec{b}[m]$$

24. Find the solution of difference equation

$$\rho_h = b_1\rho_{h-1} + b_2\rho_{h-2} + \dots + b_m\rho_{h-m}, \quad h = 1, 2, \dots$$

for the case, when the roots

$$\lambda^m = b_1\lambda^{m-1} + \dots + b_m$$

are real and different.

25. Receive the relations (1.5.9).

26. Receive the relations (1. 5.10).

27. Receive the relations

$$R^{-1}_{(k+1)} = \left\| \begin{array}{l} R^{-1}_{(k)} + R^{-1}_{(k)} J_{(k)} \vec{R}_k Q_k^{-1} \vec{R}_k^T J_{(k)} R^{-1}_{(k)} \vdots - R^{-1}_{(k)} J_{(k)} \vec{R}_k Q_k^{-1} \\ -Q_k^{-1} \vec{R}_k^T J_{(k)} R^{-1}_{(k)} \vdots Q_k^{-1} \end{array} \right\|$$

$$\tilde{R}^{-1}_{(k+1)} = \left\| \begin{array}{l} \tilde{R}^{-1}_{(k)} + \tilde{R}^{-1}_{(k)} J_{(k)} \tilde{R}_k \tilde{Q}_k^{-1} \tilde{R}_k^T J_{(k)} \tilde{R}^{-1}_{(k)} \vdots - \tilde{R}^{-1}_{(k)} J_{(k)} \tilde{R}_k \tilde{Q}_k^{-1} \\ -\tilde{Q}_k^{-1} \tilde{R}_k^T J_{(k)} \tilde{R}^{-1}_{(k)} \vdots \tilde{Q}_k^{-1} \end{array} \right\|,$$

28. Receive the relations (2.1.9).

29. Receive the relations (2.1.10).

30. Show, that for the matrix

$$B^*_k = \left\| \begin{array}{cc} 0 & B_k[k] \\ \tilde{B}_k[k] & 0 \end{array} \right\|$$

takes place the inequality $|\lambda(B^*_k)| < 1$.

31. Prove the equality

$$\det(\Phi_{(m)} - \lambda I_{(m)}) = \det(I_p \lambda^m - B_1^T[m]\lambda^{m-1} - \ldots - B_m^T[m])$$

32. Get restrictions on the correlation function of the moving average process of the second order

$$\eta_t = a_0\phi_t + a_1\phi_{t-1} + a_2\phi_{t-2}.$$

33. Show that for the process of moving average of the form

$$\eta_t = \sum_{k=-\infty}^{\infty} a_k \phi_{t-k}$$

the density has the form

$$S(\omega) = \frac{1}{2\pi} \left| \sum_{k=-\infty}^{\infty} a_k e^{ik\omega} \right|^2$$

34. Receive the relations (3.4.4).

35. Receive the relations

$$R(x_0, y_0; x_0 + x, y_0 + y) = K(x - m, y - m)R_{mn} +$$

$$+K(x - m - 1, y - n)R_{m+1,n} + K(x - m, y - m - 1)R_{m,n+1} +$$

$$+K(x - m - 1, y - n - 1)R_{m+1,n+1} = R(x, y)$$

where $m = \text{entier } (x)$, $n = \text{entier } (y)$, and

$$K(x, y) = \begin{cases} 1 - |x| - |y| + |x|\ |y|, |x| < 1,\ |y| < 1 \\ 0, |x| \geq 1,\ |y| \geq 1. \end{cases}$$

from Section 3.4.

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